3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Equations of Motion for Linear MDOF Structures
  – Apply Newton’s second law to 2-DOF system subjected to a base displacement, \( x_0(t) \)

\[
\begin{align*}
    m_1 (\ddot{x}_1 + \ddot{x}_s) + k_1 x_1 - k_2 (x_2 - x_1) &= 0 \\
    m_2 (\ddot{x}_2 + \ddot{x}_s) + k_2 (x_2 - x_1) &= 0
\end{align*}
\]

In matrix form, we write:

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{bmatrix}
+ \begin{bmatrix}
  (k_1 + k_2) & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  f \\
  J
\end{bmatrix}
\]

or

\[
[m] \ddot{\xi} + [k] \xi = -[m] \ddot{x}_s
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Equations of Motion for Linear MDOF Structures

If viscous damping is introduced in this model, it yields:

\[
[m] \ddot{x} + [c] \dot{x} + [k] x = -[m] (r) \ddot{x}_b
\]

where

- \([m]\) = global mass matrix (usually diagonal)
- \([c]\) = global damping matrix (hard to evaluate)
- \([k]\) = global stiffness matrix
- \((x), (\dot{x})\) et \((\ddot{x})\) = relative displacement, relative velocity and relative acceleration vectors
- \(\ddot{x}_b\) = absolute base acceleration
- \((r)\) = dynamic coupling vector

Note that the dynamic load vector, \((F(t)) = - (m) (r) \ddot{x}_b(t)\) in Equation (4.135)
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Equations of Motion for Linear MDOF Structures
  – Stiffness Matrix Computation
    • Evaluated by standard structural analysis techniques
  – Mass Matrix Computation
    • Lumped diagonal mass matrix
      – Simplest; mass distributed at the nodes based on tributary area
      – If more than one translating DDOF are specified for the same node (for example, vertical and horizontal), same mass is associated to each DDOF
      – Mass associated with rotating DDOF usually zero but always possible to calculate mass moment of inertia for an element of a system and to define it as a concentrated rotational mass
    • Consistent mass matrix
      – More complicated; based on energy concept
      – Used mainly for finite element dynamic analyses
      – Insure monotonic energy convergence with mesh size

• Damping Matrix Computation
  • Difficult to evaluate in an exact manner
  • Because of mathematical convenience, simple viscous damping model usually adopted
    – Coefficients of damping matrix, [c], generally defined in terms of modal damping
    – Form of damping matrix chosen in order to easily solve equations of motion.
3. “Exact” Dynamic Analysis of Linear MDOF Structures

- Equations of Motion for Linear MDOF Structures
  - Example

The mass matrix is diagonal with the same mass assigned to each DDOF.

\[
\begin{bmatrix}
m & 0 \\
0 & m
\end{bmatrix}
\]

In this system, it is easier to evaluate and invert the flexibility matrix, \([f]\), to obtain the stiffness matrix, \([k]\). To determine \([f]\), unit forces are applied in the \(x_1\) and \(x_2\) directions to get:

\[
[f] = \frac{l^3}{6EI} \begin{bmatrix}
8 & 3 \\
3 & 2
\end{bmatrix}
\]

the inverse becomes the stiffness matrix:

\[
[k] = \frac{6EI}{7l^3} \begin{bmatrix}
2 & -3 \\
-3 & 8
\end{bmatrix}
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

Natural Frequencies. The equations of motion for an undamped MDOF system in free vibrations are written:

\[ [m] \ddot{x} + [k] x = 0 \quad (4.151) \]

To solve equation 4.151, the method of separation of variables is used and the solution

\[
\begin{align*}
  x_1 &= A_1 \sin(\omega t + \phi) \\
  x_2 &= A_2 \sin(\omega t + \phi) \\
  &\vdots \\
  x_n &= A_n \sin(\omega t + \phi)
\end{align*}
\]

is:

where \( A_i \) corresponds to the amplitude of the vibrations of the \( i \)-th DDOF and \( \phi \) is the phase angle.

Substituting equation 4.153 in equation 4.151, we obtain:

\[
-\omega^2 [m] (A) \sin(\omega t + \phi) + [k] (A) \sin(\omega t + \phi) = 0
\]

This equation must be satisfied at all times \( t \), therefore:

\[
[k] (A) = 0 \quad (4.155)
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

The values $A_i$ are solved for by:

$$| (A) = [k] - \omega^2[m] | (0) = \frac{\text{co-factor} [k] - \omega^2[m]}{| [k] - \omega^2[m] |} (0)$$

In order for the system to accept non-zero values for $A_i$, the determinant of equation 4.155 must be equal to zero.

$$| [k] - \omega^2[m] | = (0)$$

If the mass matrix is diagonal, it yields:

$$\begin{vmatrix}
(k_{11} - \omega^2 m_1) & k_{12} & \cdots & k_{1N} \\
k_{21} & (k_{22} - \omega^2 m_2) & \cdots & k_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
k_{N1} & k_{N2} & \cdots & (k_{NN} - \omega^2 m_N)
\end{vmatrix} = 0$$

The values for $k_i$ and $m_i$ are known, therefore the expansion of the determinant gives a $N^\text{th}$ degree equation in $\omega^2$. This frequency equation gives $N$ real roots corresponding to $N$ natural frequencies of the system. The lowest frequency of the system is called the fundamental frequency of the system; the corresponding period (the longest) is called the fundamental period of the system.
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems
  – Example

The natural frequencies are obtained from a zero determinant:

\[
0 = \begin{vmatrix}
2 - \zeta^2 & -3 & -3 (8 - \zeta)
\end{vmatrix} = 0
\]

where

\[
\zeta = \left( \frac{7m I}{6EI} \right) \omega^2
\]

The expansion of the determinant gives a quadratic equation:

\[
\zeta^2 - 10 \zeta + 7 = 0
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

– Example

Substituting in equation 4.173, it yields:

\[ \omega_1 = 0.806 \frac{EI}{ml^2}; \quad f_1 = 0.128 \frac{EI}{ml^2}; \quad T_1 = 7.796 \frac{ml}{EI} \]

\[ \omega_2 = 2.815 \frac{EI}{ml^2}; \quad f_2 = 0.448 \frac{EI}{ml^2}; \quad T_2 = 2.232 \frac{ml}{EI} \]

Modes of Vibration (Mode Shapes). When the natural frequencies are determined, they can be substituted one by one into equation 4.155 to solve for each mode of vibration, \( (A^0) \).

\[ \begin{bmatrix} [k] - \omega^2 \begin{bmatrix} m \end{bmatrix} \end{bmatrix} (A^0) = 0 \]  \hspace{1cm} (4.177)

It is important to note that each vector \( (A^0) \) does not have an absolute value because only its shape has been determined. In fact, natural frequencies represent eigenvalues and modes of vibration represent eigenvectors of equation 4.177.
3. “Exact” Dynamic Analysis of Linear MDOF Structures

- Free Vibrations of Undamped Systems
  - Example

\[
\begin{bmatrix}
\omega_1 = 0.806 \frac{EI}{ml}
\end{bmatrix}
\]

\[
\omega_2 = 2.815 \frac{EI}{ml}
\]

Substituting for the first mode in equation 4.155, it yields:

\[
[\mathbf{k}] - \omega^2 \{m\} \mathbf{A} = 0
\]

Let us suppose:

\[
\begin{bmatrix}
A_1
\end{bmatrix} = 1
\]

then

\[
A_2 = 0.414
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

– Example

Substituting for the second mode in equation 4.155, it yields:

\[
\begin{bmatrix}
2 - 9,245 & -3 \\
-3 & 8 - 9,245
\end{bmatrix}
\begin{bmatrix}
A_2^{(0)} \\
A_3^{(0)}
\end{bmatrix} = (0)
\]

Let us suppose:

\[A_2^{(0)} = 1\]

then

\[A_3^{(0)} = -2,415\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

- Free Vibrations of Undamped Systems

**Frequency and Modal Matrices.** Once the natural frequencies of a system are calculated, they can be positioned in ascending order:

\[ \omega_1 < \omega_2 < \ldots < \omega_n \]

A frequency matrix can then be constructed:

\[
\begin{bmatrix}
\omega_1 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & \omega_2 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & \omega_3 & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \omega_i & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \ldots & \omega_n \\
\end{bmatrix}
\]

A matrix can also be constructed with the mode shapes. The columns of this modal matrix, \( [A] \), are made of modes of vibration.

\[
[A] - [A^0](A^0) \ldots (A^0) \rightarrow \text{Modal Matrix}
\]

or

\[
[A] = \begin{bmatrix}
A_1 & A_0 & A_0 \ldots A_1 - A_0^2 \\
A_0 & A_2 & A_3 \ldots A_2 - A_0^2 \\
A_0 & A_0 & A_3 \ldots A_0 - A_0^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_0 & A_0 & A_3 \ldots A_0 - A_0^2 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
A_1 & A_0 & A_0 \ldots A_1 - A_0^2 \\
\end{bmatrix}
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

**Mode Shapes Normalization.** As the mode shapes do not have absolute values, they always have to be normalized. We can suppose that the first (or the last) element of each mode shape has a unit value.

Another method to normalize a mode of vibration, \( A^{(i)} \), is by the mass matrix. Each element is normalized as such:

\[
(A^{(i)})^T [m] (A^{(i)}) = I
\]

(4.206)

This method of normalizing the modes of vibration, simplifies the computations to obtain the dynamic response of a MDOF system.

---

**Orthogonality Conditions of Mode Shapes.** For each mode of vibration \( A^{(i)} \), we have:

\[
[k] - \omega_i^2 [m] (A^{(i)}) = 0
\]

or

\[
[k] (A^{(i)}) = \omega_i^2 [m] (A^{(i)})
\]

Let us write this equation for two different modes \( A^{(r)} \) and \( A^{(s)} \) (\( r \neq s \)).

\[
[k] (A^{(r)}) = \omega_r^2 [m] (A^{(r)}) \quad (4.212)
\]

\[
[k] (A^{(s)}) = \omega_s^2 [m] (A^{(s)}) \quad (4.213)
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

Equation 4.212 can be premultiplied by \((A^o)^T\). Equation 4.213 can be transposed and postmultiplied afterward by \((A^o)^T\). It yields:

\[
\begin{align*}
(A^o)^T [k] (A^o) &= \omega^2 (A^o)^T [m] (A^o) \\
(A^o)^T [k] (A^o) &= \omega^2 (A^o)^T [m] (A^o)
\end{align*}
\] (4.214) (4.215)

But the mass matrix and the stiffness matrix are symmetrical:

\[
\begin{align*}
[k] &= [k]^T \\
[m] &= [m]^T
\end{align*}
\]

Subtracting equation 4.215 from equation 4.214, it yields:

\[
0 = (\omega^2 - \omega^2^r) (A^o)^T [m] (A^o)
\]

However, as the modes are different:

\[
\omega^2 \neq \omega^2^r
\]

we must conclude that for \(r \neq s\):

\[
(A^o)^T [m] (A^o) = 0
\] (4.219)

and

\[
(A^o)^T [k] (A^o) = 0
\] (4.220)
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems
  – Equations 4.219 and 4.220 represent orthogonality conditions of mode shapes.
  – Mode shapes are said to be normal (or orthogonal) with respect to the mass matrix and to the stiffness matrix.
  – Orthogonality conditions are very useful to:
    • simplify modal analysis
    • verify computation of mode shapes;
    • simplify mode shapes computation by using other techniques (Chopra, 2001).

Moreover, if all mode shapes are normalized with respect to the mass matrix (equation 4.206), it yields:

\[
[A^T \{m\} \{A\}] = [I]
\]

(4.221)

where \([I]\) is the identity matrix.

It also yields:

\[
[A^T \{k\} \{A\}] = \{\omega^2\}
\]

(4.222)

If equations 4.221 and 4.222 are satisfied, the modes are said to be orthonormal.
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

Rayleigh’s Method. Lord Rayleigh (1945) developed a procedure to analyse vibrating systems based on the conservation of energy. This technique is mostly used in earthquake engineering to estimate the fundamental period of vibration of a structure.

To develop Rayleigh’s formula, we use the equations of motion of an undamped MDOF system in free vibrations (equation 4.151). This equation is premultiplied by the transpose of the velocity vector \( \{\dot{x}\}^T \) and is integrated with respect to time.

\[
\int_0^t (\dot{x} \dot{x}^T \{m\} \{\dot{x}\}) dt + \int_0^t (\dot{x} \dot{x}^T \{k\} \{x\}) dt = \text{constant} \tag{4.223}
\]

Now, differential equations are used between the displacement, the velocity and the acceleration.

\[
\begin{align*}
\ddot{x} & = \frac{dx}{dt} \\
\dot{x} & = \frac{d\dot{x}}{dt} \tag{4.224}
\end{align*}
\]

Substituting equation 4.224 into equation 4.223, it yields:

The two terms on the left-hand side of this equation can be integrated.

\[
\int_0^t \frac{1}{2} (\dot{x} \dot{x}^T \{m\} \{x\}) dt + \int_0^t \frac{1}{2} (\dot{x} \dot{x}^T \{k\} \{x\}) dt = \text{constant} \tag{4.225}
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

\[
\frac{1}{2} (\ddot{x})^T [m] \ddot{x} + \frac{1}{2} (x)^T [k] x = \text{constant} \quad (4.226)
\]

The first term on the left-hand side represents the kinetic energy of the system at time \( t \). The second term on the left-hand side represents the elastic strain energy of the system at time \( t \). Equation 4.226 reflects the law of energy conservation of an elastic, undamped system. At all times, the energy in the system is equal to a constant.

Now, let us suppose that a system vibrates in a specific mode \( (A^0) \) with a corresponding natural frequency \( \omega_i \). The system’s response can be expressed by:

\[
\begin{align*}
(x) &= (A^0) \sin (\omega_i t + \phi) \\
(\dot{x}) &= (A^0) \omega_i \cos (\omega_i t + \phi)
\end{align*}
\]

(4.227)

Substituting equation 4.227 into equation 4.226, it yields:

\[
\frac{1}{2} (A^0)^T [m] (A^0) \sin^2 (\omega_i t + \phi) + \frac{1}{2} (A^0)^T [k] (A^0) = \text{constant} \quad (4.228)
\]
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

\[ \frac{1}{2} \omega^2 \cos^2(\omega t + \phi)(A^0)^2[m](A^0) + \frac{1}{2} \sin^2(\omega t + \phi)(A^0)^2[k](A^0) = \text{constant} \quad (4.228) \]

In equation 4.228, we notice that the kinetic energy is zero \((\cos^2(\omega t + \phi) = 0)\) when the strain energy is maximum \((\sin^2(\omega t + \phi) = 1)\) and that the kinetic energy is maximum \((\cos^2(\omega t + \phi) = 1)\) when the strain energy is zero \((\sin^2(\omega t + \phi) = 0)\). As the total energy in the system remains constant, the maximum kinetic energy must be equal to the maximum strain energy.

\[ \frac{1}{2} \omega^2 (A^0)^2[m](A^0) = \frac{1}{2} (A^0)^2[k](A^0) \]

The natural frequency can be isolated and an equation for the natural period of vibration can be found.

\[ \omega_n = \sqrt{\frac{(A^0)^2[k](A^0)}{(A^0)^2[m](A^0)}} \]

\[ T_i = 2\pi \sqrt{\frac{(A^0)^2[m](A^0)}{(A^0)^2[k](A^0)}} \]

Knowing the exact mode of vibration \((A^0)\), then the exact corresponding natural period \(T_i\) can be calculated. However, if the mode shape is only approximated, then the period of vibration is an estimate.
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

Generally, the first mode of vibration of a system is estimated by static deformations, \( \delta \), generated by a careful selection of the static loads, \( \{F\} \).

\[
\{F\} = [k] \{\delta\}
\]

\[
\{\delta\} \approx \{\delta\}
\]

The estimate of the fundamental period becomes:

\[
T_1 \approx 2\pi \sqrt{\frac{\sum_{j=1}^{N} W_j \delta_j^2}{g \sum_{j=1}^{N} F_j \delta_j}}
\]

where \( W_j \) = weight associated to the jth degree-of-freedom
\( N \) = number of degrees-of-freedom
\( g \) = acceleration of gravity
3. “Exact” Dynamic Analysis of Linear MDOF Structures

• Free Vibrations of Undamped Systems

Often, the fundamental mode shape can be estimated, using static loads equal to the weight associated to the degrees-of-freedom \((F_j = W_j)\).

\[
T_j \approx 2\pi \sqrt{\frac{\sum_{j=1}^{N} W_j \delta_j^2}{\sum_{j=1}^{N} W_j \delta_j}}
\]

Equation 4.235 is Rayleigh’s formula