Lecture 06 – Slope Deflection – Displacement Method

1. Formulation
2. Reduced order models
3. Examples

References:  
Leet & Uang
Hieberl
Slope Deflection Method

In indeterminate structures define rotations in either fixed supports or at continuity joints, as unknowns (not forces). Express the rotations in terms of internal forces and reactions, use the equilibrium to provide additional equations to solve indeterminacy.

**Example**

**This structure is five times indeterminate:**

R = 8  e = 3  c = 5.

However, neglecting the axial forces, the indeterminacy reduces to three:

Moreover, using kinematic unknowns solution may reduce to only two equations.

Knowing that $\theta_A = \theta_B = \phi$ due to fixity, only two unknowns remain.

Writing equations of equilibrium of moments in B and C, two equations can be developed.
Assume a span in a continuous beam as shown below, which does not have translations at the ends, but only with end forces, in equilibrium. It is possible to formulate relations between the end forces, the loads and the end rotations (no vertical displacements):

\[
\text{All end forces are expressed as function of displacements and rotations at the ends.}
\]

\[
\begin{align*}
\text{(1)} & \quad M_{A0} = \frac{A_{Mw} \cdot Y_2}{E_I} \\
\text{(2)} & \quad M_{BA} = \frac{M_{AB} \cdot L}{2} - \frac{2}{3E_I} \\
\text{(3)} & \quad M_{BA} = \frac{L^2}{2} \cdot \frac{1}{3E_I} \\
\text{(4)} & \quad \theta_A = \frac{M_{AB} \cdot L}{3E_I} \\
\end{align*}
\]
Therefore

\[ \Theta_a = \left( \frac{A h w - 1}{2Ei} \right) + \frac{M_{ab} L}{3Ei} + \frac{M_{bc} L}{6Ei} \]

\[ \Theta_b = \left( \frac{A h w + 1}{2Ei} \right) + \frac{M_{ab} L}{6Ei} + \frac{M_{bc} L}{3Ei} \]

\[ \Theta_c = \Theta_{bc} + \frac{M_{ab} L}{6Ei} + \frac{M_{bc} L}{3Ei} \]

\[ \Theta_a = \Theta_{bc} - \frac{M_{bc} L}{3Ei} - \frac{M_{ab} L}{6Ei} \]

\[ M_{ab} = -\frac{2Ei}{L} (\Theta_a + \Theta_b) + \frac{2Ei}{L} (\Theta_{bc} + \Theta_{bc}) \]

\[ M_{bc} = \frac{2Ei}{L} (\Theta_a + \Theta_b) + \frac{2Ei}{L} (\Theta_{bc} + \Theta_{bc}) \]
Assuming that the supports can move vertically (or transverse to the beam) then we can add the influence of such transverse movement as a rotation of the baseline:

\[
\theta_a = \theta_a - \delta_y^A \quad \text{and} \quad \theta_B = \theta_B - \delta_y^B.
\]

Assume that \( M_{AB} \) is acting also clockwise; this changes all signs in first equation:

Then the end moments:

\[
M_{AB} = \frac{2EI'}{L} \left( 2\theta_a + 2\theta_B - 3\delta_y^A \right) - \frac{2EI'}{L} \left( 2\theta_{ac} + \theta_{bc} \right) - F_E M_A
\]

\[
M_{AC} = \frac{2EI'}{L} \left( \theta_a + 2\theta_B - 3\delta_y^A \right) - \frac{2EI'}{L} \left( \theta_{ac} + 2\theta_{bc} \right) - F_E M_B.
\]
Chea = reaction moments in the end supports = Fixed End Moments (equal Plas for 0 = 0.3 = 0).

Therefore: \( M_{A5} = \frac{2EI}{L} \left( \theta_A + \theta_F - 3\phi_{AB} \right) + FEM_{A5} \)

\( M_{BA} = \frac{2EI}{L} \left( \theta_A + 2\theta_B - 3\phi_{AB} \right) + FEM_{BA} \)

Ok:

\[
\begin{align*}
M_{NF} &= \frac{2EI}{L} \left( \theta_N + \theta_F - 3\phi_{NF} \right) + FEM_{NF} \\
N &= N_{NEW} \\
F &= F_{NEW}
\end{align*}
\]

where \( FEM_{NF} = -\frac{2EI}{L} \left( 2\phi_{NF} + \theta_{FL} \right) \)

direction of all moments and rotations.
Example of fixed end moments

Moments that produce zero rotation at ends:

\[ FEM_{AB} = \frac{2EI}{L} (2\theta_{AL} + \theta_{BL}) \]

\[ \theta_{AL} = \left(\frac{PL}{4}\right) \cdot \frac{1}{2EI} = \frac{PL^2}{16EI} \]

\[ \theta_{BL} = \left(\frac{PL}{4}\right) \cdot \frac{1}{2EI} = \frac{PL^2}{16EI} \]

\[ FEM_{AB} = -\frac{2EI}{L} \left(2 \cdot \frac{PL^2}{16EI} - \frac{PL^2}{16EI}\right) = -\frac{PL}{8} \]

\[ FEM_{BA} = \frac{2EI}{L} (\theta_{BL} + 2\theta_{AL}) \]

or:

\[ \frac{2EI}{L} (2\theta_{BL} + \theta_{AL}) \]

\[ FEM_{BA} = -\frac{2EI}{L} \left(-\frac{PL^2}{16EI} \cdot 2 + \frac{PL^2}{16EI}\right) = \frac{PL}{8} \]
Fixed-End Moments

From Leet and Uang (2nd Edition)
Note: For a span in which moment at end is zero:

\[ \begin{aligned}
\phi_{PA} &= \varphi \\
\phi_{PB} &= \frac{2EI}{L} \left( \frac{\varphi_B}{2} + \frac{\varphi_A}{2} \right) + \text{FEM}_{BA} = \varphi \\
\phi_B &= -\frac{E}{2} \\
M_{AB} &= \frac{E}{2a} \left[ 2\varphi_A - \frac{\varphi_B}{2} \right] + \text{FEM}_{AB} \\
M_{BA} &= \frac{3EI}{P} \varphi_A + \text{FEM}_{BS}
\end{aligned} \]
Score deflection technique

1. Determine number of joint rotations.
2. Determine the end moments for each member:
   \[ M_{NF} = \frac{2E}{h_{NF}} \left( 2\theta_B + \theta_F - 2\theta_C \right) + \frac{PL}{A} \]
   \[ C_{M_{BA}} = \frac{2E}{h_{BA}} \left( 2\theta_B + \theta_C - \frac{w_1}{12} \right) - \frac{PL}{A} \]
   \[ C_{M_{BC}} = \frac{2E}{h_{BC}} \left( 2\theta_B + \theta_C - \frac{w_1}{12} \right) - \frac{PL}{A} \]
3. Write equation of equilibrium for moments in each joint with free movement:
   \[ M_{BA} + M_{BC} = 0 \]
   \[ M_{CB} + M_{CD} = 0 \]