Hysteresic Models for Deteriorating Inelastic Structures

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**Abstract:** The modeling of deteriorating hysteretic behavior is becoming increasingly important, especially in the context of seismic analysis and design. This paper presents the development of a versatile smooth hysteretic model based on internal variables, with stiffness and strength deterioration and with pinching characteristics. The theoretical background, development, and implementation of the model are discussed. Examples are shown to illustrate the features of the model. Many inelastic constitutive models in popular use have been developed independently of each other based on different behavioral, physical, or mathematical motivations. This paper attempts to unify the concepts underlying such models. Such a holistic understanding is essential to realize limitations in application of inelastic models and to extend 1D models to 3D models featuring interaction between various stress results.

**Introduction**

Hysteresis is a highly nonlinear phenomenon occurring in many disciplines involving systems that possess memory, including inelasticity, electricity, magnetism etc. Structures subjected to strong earthquake excitation are designed to dissipate energy by inelastic material behavior, interface friction, etc. However, under repeated cyclic deformation, there is invariably deterioration in the characteristics of such hysteretic behavior. Such deterioration must be taken into account in the modeling and design of seismic-resistant structural systems. The basic requirement to perform such analyses is the availability of accurate constitutive models capable of representing deteriorating structural behavior.

Several hysteretic models have been developed. These can be broadly classified into two types, polygonal hysteretic model (PHM) and smooth hysteretic model (SHM). Models based on piecewise linear behavior are PHMs. Such models are most often motivated by actual behavioral stages of an element or structure, such as initial or elastic, cracking, yielding, stiffness and strength degrading stages, and crack and gap closures. Examples include Clough’s model (Clough 1966), Takeda’s model (Takeda et al. 1970), and the Park’s “three-parameters” model (Park et al. 1987). Sivaselvan and Reinhorn (1999) presented a detailed description of a general framework for PHMs. On the other hand, SHMs refer to models with continuous change of stiffness due to yielding but sharp changes due to unloading and deteriorating behavior. The Wen-Bouc model (Bouc 1967; Wen 1976) and Ozdemir’s model (Ozdemir 1976) are some examples of SHMs. Thyagarajan (1989) discussed a discrete element model for hysteretic behavior based on the concept proposed by Iwan (1966). This is a polygonal model that becomes smooth in the limit of infinite elements. Mostaghel (1999) presented a differential equation description of a class of PHMs with deteriorating characteristics.

Many of these models that are in popular use have been developed independently of each other based on different behavioral, physical, or mathematical motivations. However, a closer examination presented herein would show that they share several features and stem from a common theoretical base. Such an understanding is important in developing new models and in recognizing physical limitations of many existing models.

The objectives of the work reported here are twofold: (1) To present versatile SHM, with stiffness and strength deterioration and pinching characteristics, derived from inelastic material behavior; and (2) to present a holistic picture of the modeling of ID inelastic material behavior and justify the use of such models to represent the relationships between stress results and strains.

The SHM presented herein was developed in the context of moment-curvature relationships of beam-columns. Therefore the stress variable, or force, is here referred to as “moment” (\(M\)) and the strain variable, or deformation, as “curvature” (\(\phi\)). However, these can be replaced by any other work-conjugate pair, according to the application under consideration.

**SHM**

The smooth model discussed here is a variation of the model originally proposed by Bouc (1967) and modified by several others (Wen 1976; Baber and Noori 1985; Casciati 1989; Reinhorn et al. 1995). The derivation of this model from the theory of viscoplasticity and its resemblance to the endochronic constitutive theory are discussed in a subsequent section.

**Plain Hysteretic Behavior without Degradation**

Plain hysteretic behavior with postyielding hardening can be modeled using two springs (Springs 1 and 2 of Fig. 1). One of the springs is linear elastic at all deformations, and the second changes stiffness upon yield. The two springs undergo the same deformation under a bending moment (or generalized force \(M\)). The springs share the moment proportionally to their instantaneous stiffnesses. The portion of the moment shared by the hysteretic spring is denoted by \(M^*\). The combined stiffness is given by

\[
K = K_{\text{postyield}} + K_{\text{hysteretic}} \quad (1)
\]

**Spring 1: Postyield Spring**

A linear elastic spring represents the postyielding stiffness

\[
K_{\text{postyield}} = aK_0 \quad (2)
\]

where \(K_0\) is total initial stiffness (elastic); and \(a\) is ratio of postyielding to initial stiffness ratio.

**Spring 2: Hysteretic Spring**

This purely elastoplastic spring has a smooth transition from the elastic to the inelastic range displaying degradation phenomena. The nondegrading stiffness is
Hysteretic Behavior with Degradation

Often structures that undergo inelastic deformations and cyclic behavior weaken and lose some of their stiffness and strength. Moreover, often gaps develop due to cracking and the material becomes discontinuous. The hysteretic model developed herein can accommodate changes in the stiffness, strength, and pinching due to gap opening and closing.

Stiffness Degradation

Stiffness degradation occurs due to geometric effects. The elastic stiffness degrades with increasing ductility. It has been found empirically that the stiffness degradation can be accurately modeled by the pivot rule (Park et al. 1987). According to this rule, the load-reversal branches are assumed to target a pivot point on the initial elastic branch at a distance of \( \alpha M_i \) on the opposite side, where \( \alpha \) is the stiffness degradation parameter. This is shown in Fig. 2(a). From the geometry in Fig. 2(a), the stiffness degradation factor is given by

\[
K_{cur} = R_a K_0 = \frac{M_{cur} + \alpha M_i}{K_i \phi_{cur} + \alpha M_i} K_0
\]

where \( M_{cur} \) is the current moment; \( \phi_{cur} \) is current curvature; \( K_0 \) is initial elastic stiffness; \( \alpha \) is stiffness degradation parameter; and \( M_i = M_i^* \) if \( (M_{cur}, \phi_{cur}) \) is on the right side of the initial elastic branch and \( M_i = M_i^- \) if \( (M_{cur}, \phi_{cur}) \) is on the left side. However, since stiffness degradation occurs only in the hysteretic spring, only the hysteretic stiffness is modified and is given by

\[
K_{hyst} = (R_a - a) K_0 \left\{ 1 - \left[ \frac{M^*}{M_i^*} \right]^N \{ \eta_1 \ \text{sgn}(M^* \phi) + \eta_2 \} \right\}
\]

Ranges of variation of \( \alpha \) indicate that for large values (\( \alpha > 200 \)), no deterioration occurs, whereas small values (\( \alpha < 10 \)) produce substantial degradation (Sivaselvan and Reinhorn 1999).

Strength Degradation

Strength degradation is modeled by reducing the capacity in the backbone curve, as shown schematically in Fig. 2(b). Mathematically, this is equivalent to specifying an evolution equation for the yield moment. The strength degradation rule can be formulated to include an envelope degradation, which occurs when the maximum deformation attained in the past is exceeded, and a continuous energy-based degradation. The rule reads

\[
M_i^{*+} = M_i^{*0} \left[ 1 - \left( \frac{\phi_{cur}}{\phi_{max}} \right)^{1/\beta_2} \right] \left[ 1 - \frac{\beta_2}{1 - \beta_2} \right] H_{max}
\]

where \( M_i^{*+} \) is positive or negative yield moment; \( M_i^{*0} \) is initial positive or negative yield moment; \( \phi_{cur} \) is maximum positive or negative curvatures; \( \phi_{max} \) is positive or negative ultimate curvatures; \( H \) is hysteretic energy dissipated, obtained by integrating the hysteretic energy quotient; \( H_{max} \) is hysteretic energy dissipated when loaded monotonically to the ultimate curvature without any degradation; \( \beta_2 \) is ductility-based strength degradation parameter; and \( \beta_2 \) is energy-based strength degradation parameter. The second term on the right-hand side of (8) represents strength degradation due to increased deformation, and the third term represents strength degradation due to hysteretic energy dissipated. The quotient of the hysteretic energy in incremental form is given by

\[
\Delta H = \left[ M + (M + \Delta M) \right] \left( \Delta \phi - \frac{\Delta M}{K_i K_0} \right)
\]

The differential equations governing strength degradation in the SHM can be obtained by differentiating (8)
\[ \frac{dM^{s/f}}{dt} = M^{s/f}_0 \left\{ 1 - \frac{\beta_s}{1 - \beta_s} H \right\} \]

\[ \cdot \left[ \frac{1}{\beta_s (\phi_s^{s/f})^{\gamma_s}} \left( \phi_s^{s/f} \right)^{\gamma_s-1} \right] \phi_s^{s/f} \]

\[ + \left[ 1 - \left( \frac{\phi_{\text{max}}^{s/f}}{\phi_s^{s/f}} \right)^{1/\gamma_s} \right] \left\{ -\frac{\beta_s}{1 - \beta_s} H \right\} \] \hspace{1cm} (10)

Eq. (10) requires the hysteretic energy quotient in rate form

\[ H = M \left( \phi - \frac{M}{R_s K_s} \right) = M \left[ 1 - \left( K_{\text{yield}} + R_s K_{\text{hysteretic}} \right) \right] \] \hspace{1cm} (11)

The evolution equations for the maximum positive and negative curvatures can be written

\[ \dot{\phi}_{\text{max}} = \dot{U}(\phi - \phi_{\text{max}}) U(\phi); \quad \dot{\phi}_{\text{max}} = \dot{U}(\phi_{\text{max}} - \phi)(1 - U(\phi)) \] \hspace{1cm} (12)

where \( U(x) \) is the heaviside step function. The differential equations [(10)–(12)] govern strength degradation.

**Pinching or Slip**

Pinched hysteretic loops usually are a result of crack closure, bolt slip, etc. An additional spring (Spring 3 of Fig. 1) called the slip-lock spring (Baber and Noori 1985; Reinhorn et al. 1995) is added in series to the hysteretic spring to model this effect. The stiffness of the slip-lock spring can be written

\[ K_{\text{slip-lock}} = \left\{ \frac{s}{\pi M^s} \right\} \exp \left\{ -\frac{1}{2} \left( \frac{M^s - M^s_*}{M^s} \right)^2 \right\} \] \hspace{1cm} (13)

where \( s = \) slip length; \( R_s (\phi_{\text{max}} - \phi_{\text{max}}); M^s = \sigma M^s = \) measure of the moment range over which slip occurs; \( M^s_* = \lambda M^s = \) mean moment level on either side about which slip occurs; \( \sigma, \lambda, \) and \( \lambda \) are parameters of the model; and \( \phi_{\text{max}} \) and \( \phi_{\text{max}} \) are maximum curvatures reached on the positive and negative sides, respectively, during the response. The variation of the flexibility of the slip-lock element can be chosen as Gaussian or any other distribution provided that \( \int_{-\infty}^{+\infty} (K_{\text{slip-lock}})^{-1} dM = s \) the slip length. The stiffness of the combined system is then given by

\[ K = K_{\text{yield}} + \frac{K_{\text{hysteretic}} K_{\text{slip-lock}}}{K_{\text{slip-lock}} + K_{\text{hysteretic}}} \] \hspace{1cm} (14)

**Gap-Closing Behavior**

Often, hysteretic elements exhibit stiffening under higher deformations. This happens, for example, in metallic dampers (Soong and Dargush 1997) when axial behavior predominates bending behavior and in bridge isolators (Reichman and Reinhorn 1995) due to closing of the expansion joints. Such behavior can be modeled by introducing an additional gap-closing spring in parallel, as shown in Fig. 1. The internal moment and the stiffness of this spring are given by

\[ M^{s/g} = \kappa N_{\text{gap}} (\phi - \phi_{\text{gap}})^{\gamma_{\text{gap}}-1} U(\phi - \phi_{\text{gap}}) \] \hspace{1cm} (15)

\[ K_{\text{gap-closing}} = \kappa N_{\text{gap}} (\phi - \phi_{\text{gap}})^{\gamma_{\text{gap}}-1} U(\phi - \phi_{\text{gap}}) \] \hspace{1cm} (16)

where \( M^{s/g} = \) moment in the gap-closing spring; \( K_{\text{gap-closing}} = \) stiffness of the gap-closing spring; \( \phi_{\text{gap}} = \) gap-closing curvature; \( U = \) heaviside step function; and \( \kappa \) and \( N_{\text{gap}} \) are parameters.

**Numerical Solution**

The equations governing the SHM may be solved by (1) the conventional displacement-based incremental approach; and (2) the state-space approach (Simeonov 1999). Using the latter solution approach, (5) and (10)–(12) can be solved simultaneously with the other governing equations of the structure. However, only the former approach will be discussed here. For this purpose, (5) and (10)–(12) have to be written in time-independent form with displacement/curvature as the independent variable. Also, since the postyielding and gap-closing springs are algebraic, only the hysteretic and slip-lock springs are solved and the results added. This produces the following time-independent differential equations within a global time step

\[ \frac{dM^s}{d\phi} = \frac{K_{\text{hysteretic}} K_{\text{slip-lock}}}{K_{\text{slip-lock}} + K_{\text{hysteretic}}} \] \hspace{1cm} (17a)

\[ \frac{dM^s}{d\phi} = \frac{M^s_0}{M^s} \left\{ 1 - \frac{\beta_s}{1 - \beta_s} H \right\} \]

\[ \cdot \left[ \frac{1}{\beta_s (\phi_s^{s/f})^{\gamma_s}} \left( \phi_s^{s/f} \right)^{\gamma_s-1} \right] \phi_s^{s/f} \]

\[ + \left[ 1 - \left( \frac{\phi_{\text{max}}^{s/f}}{\phi_s^{s/f}} \right)^{1/\gamma_s} \right] \left\{ -\frac{\beta_s}{1 - \beta_s} H \right\} \] \hspace{1cm} (17b)

\[ \frac{d\phi_{\text{max}}}{d\phi} = \frac{U(\phi - \phi_{\text{max}}) U(\Delta\phi); \quad \frac{d\phi_{\text{max}}}{d\phi} = U(\phi_{\text{max}} - \phi)(1 - U(\Delta\phi)) \] \hspace{1cm} (17c)

\[ \frac{dH}{d\phi} = M^s \left( 1 - \frac{1}{(1 - \beta_s) R_s K_s} \right) \] \hspace{1cm} (17d)

Eq. (17) can be solved within each global integration step using any method such as the RK45 or the semi-implicit Rosenbrock methods (Nagarajah et al. 1989). Sivasethyan and Reinhorn (1999) presented an approximate iteration matrix for use with the implicit methods.

**Numerical Examples**

Examples of various types of hysteretic behavior modeled by SHM are shown in Fig. 3. Ranges of degradation parameters obtained from experimental results of the SAC Joint Venture (1996) are shown in Table 1. The identification of the hysteretic parameters was done using a standard least-square fit. The comparison between behavior predicted by SHM and

![Examples of Hystereses (Moment versus Curvature Ductility)](image)
the experimental results is presented in Fig. 4 and shows good approximations.

OTHER DEGRADATION RULES

The degradation of the nonlinear spring discussed above may also be produced by other rules. The general procedure, however, is the same irrespective of the particular rule. A few examples are presented below:

Mostaghel’s Stiffness Degradation (Mostaghel 1999)

\[ R_K = (1 + \alpha H)^{-1} \quad \text{or} \quad R_K = e^{-\alpha H} \]

where \( H \) is the hysteretic energy and \( \alpha \) is the stiffness degradation parameter.

Mostaghel’s Strength Degradation

\[ M_{y,\text{degr}} = M_{y,\text{init}}(1 + \beta H)^{-\gamma} \quad \text{or} \quad M_{y,\text{degr}} = M_{y,\text{init}} e^{-\beta H} \]

where \( \beta \) is the strength degradation parameter.

Ozdemir’s Stiffness Degradation (Ozdemir 1976)

\[ R_K = g(R_K) \left| \frac{M}{M_{\text{nom}}^{\gamma}} \right| |\phi| \quad \text{or} \quad R_K = -\alpha \left| \frac{M}{M_{\text{nom}}^{\gamma}} \right| |\phi| \quad (18) \]

where \( \alpha \) = stiffness degradation parameter.

Ozdemir’s Strength Degradation

The yield moment is reduced according to the moment ratio

\[ M_{y,\text{degr}} = f(M_{y,\text{init}}) \left| \frac{M}{M_{\text{nom}}^{\gamma}} \right| |\phi| \quad \text{or} \quad M_{y,\text{degr}} = -\beta \left| \frac{M}{M_{\text{nom}}^{\gamma}} \right| |\phi| \quad (19) \]

where \( \beta \) = strength degradation parameter.

REPRESENTATION OF INELASTIC MATERIAL BEHAVIOR IN ONE DIMENSION

This section attempts to indicate the basis of the formulation presented in the previous section through a unified approach to various hysteretic models. In an inelastic material, the strain at any point is not determined by the current stress alone, but also by the history of loading. There are many theories to represent inelastic material behavior, such as the theory of materials with memory, the internal state variable theory, and

<table>
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<th>SPECIMEN</th>
<th>EXPERIMENTAL RESPONSE</th>
<th>ANALYTICAL RESPONSE</th>
<th>PARAMETERS</th>
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</thead>
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<td>EERC-RN3</td>
<td><img src="image1.png" alt="EERC-RN3" /></td>
<td><img src="image2.png" alt="EERC-RN3" /></td>
<td>( \alpha = 1.0 )</td>
</tr>
<tr>
<td>UCSD-1R</td>
<td><img src="image3.png" alt="UCSD-1R" /></td>
<td><img src="image4.png" alt="UCSD-1R" /></td>
<td>( \beta_1 = 0.60 )</td>
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<tr>
<td>UTA-3R</td>
<td><img src="image5.png" alt="UTA-3R" /></td>
<td><img src="image6.png" alt="UTA-3R" /></td>
<td>( \beta_1 = 0.40 ), ( \mu_{\text{init}} = 7.1 )</td>
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FIG. 4. Results from Connection Tests (SAC Joint Venture 1996)
physically motivated models. Sivaselvan and Reinhold (1999) presented a detailed schematic of the interrelationships between these. A few of these models are discussed here to demonstrate the conceptual development.

**ALGEBRAIC MODELS**

Algebraic nonlinear models such as the Ramberg-Osgood model, the Menegotto-Pinto model, etc., have been extended to hysteretic behavior using Masing’s Hypothesis; models based on differential equations, that exhibit smooth virgin curves generally exhibit local violation of Drucker’s stability postulate (Thyagarajan 1989). The algebraic models exhibit smooth virgin curves and yet do not violate Drucker’s stability postulate. However, their implementation is extremely involved.

*Masing’s Hypothesis (Masing 1926)*

If the virgin loading curve is given by an algebraic relationship, \( f(\phi, M) = 0 \), then, according to Masing’s hypothesis, the curve between the vertex ver points (\( \phi_{Cer}, M_{Ver} \)) and (\( \phi_{Mer}, M_{Mer} \)) is given by

\[
f(HM - M_{Ver}^k, \phi - \phi_{Ver}) = 0
\]

(Ramberg-Osgood Model (Ramberg and Osgood 1943))

\[
\frac{\phi}{\phi_r} = \frac{M}{M_r} \left( 1 - \left[ \frac{M}{M_r} \right]^N \right)
\]

where \( \eta \) (positive) and \( N \) (odd integer) = parameters of the model.

*Menegotto-Pinto Model (Menegotto and Pinto 1973)*

\[
M = \phi KU(\phi)U(M)U(M_r - M) + \phi KU(\phi)U(-M)
\]

+ \( \phi KU(\phi)U(M)U(M_r + M) \) (23a)

or

\[
M = K \left[ 1 - \frac{1 + \text{sgn}(M\phi)}{2} \right] \left( U(M_r - M)U(M)
\]

+ \( U(-M_r + M)U(-M) \) \( \phi \) (23b)

where \( U(\ ) = \text{step function} \). Eqs. (23a) and (23b) can be shown to be exactly equivalent.

**Plasticity/Viscoplasticity without Yield Surface**

Several constitutive models for both rate-dependent (viscoplastic) and rate-independent (plastic) inelastic behavior have been derived without a formal hypothesis of a yield surface. Three such models are discussed here: (1) The endochronic model; (2) the Ozdemir model; and (3) the Wen-Bouc model. Although developed based on different motivations, all three models exhibit similar behavior and become identical under certain limiting conditions.

*Endochronic Theory*

The basic concept (Valanis 1971, 1980) of the endochronic theory is that of an intrinsic time \( \tau \) that is related to the deformation history of the material (the relation itself being a material property). This is then used in place of real time in the convolution integral of linear viscoelasticity. Bazant and Bhat (1976) provided a physical interpretation of the endochronic theory for the 1D case starting from the Maxwell model and replacing real time by a function of plastic strain. The resulting constitutive equation is

\[
M = K \phi - \frac{1}{Z} \frac{M}{M(\phi)}
\]

where \( Z = M/K_c \).

*Ozdemir Model (Ozdemir 1976)*

The systematic development of the Ozdemir model by the generalization of the linear dashpot is shown in Fig. 5, where \( N \) is an odd integer to preserve signs. Adding a spring in series to form a Maxwell element with a nonlinear dashpot results in

\[
\phi = \frac{M}{K_0} + \frac{1}{\tau} M^\eta \text{ or } \frac{\phi}{\phi_r} = \frac{M}{M_r} + \frac{1}{\tau} \left( \frac{M}{M_r} \right)^\eta
\]

or

\[
\frac{M}{M_r} = \frac{\phi}{\phi_r} - \frac{1}{\tau} \left[ \frac{M}{M_r} \right]^\eta
\]

where \( \tau = c M_r^\eta = \phi_r = \text{the time constant. As } N \to \infty \), the model approaches an elastoplastic model. This is the Ozdemir model. However, the yield strength is rate-dependent. Ozdemir (1976) showed that a time-constant \( \tau = [\phi/\phi_r] \) exists for each loading rate \( \phi \) that makes the model rate independent. The rate-independent Ozdemir model is then

\[
M = \phi K U(\phi) U(M) U(M_r - M) + \phi K U(\phi) U(-M)
\]

+ \( \phi K U(\phi) U(M) U(M_r + M) \) (25)

or

\[
M = \phi K U(\phi) U(M) U(M_r - M) + \phi K U(\phi) U(-M)
\]

+ \( \phi K U(\phi) U(M) U(M_r + M) \) (26)

since \( N \) is odd. The nonlinear dashpot can now be replaced by a slider. If (25) is considered to be governed by a material time \( T \) rather than the inertial time \( t \). Then

\[
M = \phi K U(\phi) U(M) U(M_r - M) + \phi K U(\phi) U(-M)
\]

+ \( \phi K U(\phi) U(M) U(M_r + M) \) (27)

Comparing (27) with (26), we have

\[
\frac{1}{\tau} \frac{dT}{dt} = \frac{1}{|\phi_r|} \frac{d[\phi]}{dt} \text{ or } \frac{dT}{\tau} = \frac{d[\phi]}{|\phi_r|}
\]

where \( T \) = now similar to the intrinsic time of Bazant and Valanis. Hardening may be introduced into the model by adding a linear hardening spring in parallel with the nonlinear dashpot as shown in Fig. 5(e). The force in this spring is referred to as the back-stress \( S \).

*Reformulated Wen-Bouc Model*

Bouc (1967) and Wen (1976), using some mathematical reasoning, came up with (3), as reformulated by Reinhold et al. (1995). It can be shown, however, that when \( a = 0 \) (i.e., elastic-ideal plastic), this equation can be derived from the equations of 1D plasticity based on a yield surface by smoothing the
heaviside step function. Starting from (23b) and assuming that $M_r^* = -M_r^* = |M_r|$, we have

$$\frac{\dot{M}}{M_r} = \frac{\dot{\phi}}{\phi_r} - \frac{1}{\tau} \left( \frac{M}{M_r} \right)^N$$

The heaviside step function can be smoothened as follows when $|M| \leq |M_r|:

$$U(|M_r| - |M|) \approx \left( \frac{M}{M_r} \right)^N$$

Substituting (30) into (29) leads to a special case of (3) for which $\eta_1 = \eta_2 = 0.5$. When $M$ is maximum

$$\frac{dM}{d\phi} \bigg|_{M=M_{\text{max}}} = K \left\{ 1 - \left( \frac{M}{M_r} \right)^N \left[ \eta_1 + \eta_2 \right] \right\} = 0$$

or

$$M_{\text{max}} = M_r \left( \frac{1}{\eta_1 + \eta_2} \right)^{1/\eta_N}$$

Therefore for such a case $\eta_1 + \eta_2 = 1$ always, as shown by Constantinou and Adnane (1987). A more general derivation of the relationship between a 3D Wen-Bouc model and a plasticity model based on a yield function is given by Casciati (1989). Let us now consider two limiting cases of the reformulated Wen-Bouc model:

- Case 1: $\eta_1 = 1; \eta_2 = 0$. If $N$ is an odd integer, then

$$M = K \left\{ 1 - \left( \frac{M}{M_r} \right)^N \left[ \eta_1 + \eta_2 \right] \right\} \phi = K \left\{ \phi - |\phi| \left( \frac{M}{M_r} \right)^N \right\}$$

or

$$\frac{M}{M_r} = \frac{\dot{\phi}}{\phi_r} - \frac{1}{\tau} \left( \frac{M}{M_r} \right)^N$$

which is exactly the Ozdemir model of (26).

- Case 2: $\eta_1 = 0; \eta_2 = 1$

$$M = K \left\{ 1 - \left( \frac{M}{M_r} \right)^N \right\} \phi \quad \text{or} \quad \phi = \frac{M}{K} \left\{ 1 - \left( \frac{M}{M_r} \right)^N \right\}^{-1}$$

which is the Menegotto-Pinto equation [(22)] for nonlinear elastic behavior. The Wen-Bouc model can therefore be thought of as a weighted combination of the rate-independent Ozdemir model and the nonlinear elastic Menegotto-Pinto model, which produces a smooth elastoplastic model for the particular weight set $\eta_1 = \eta_2 = 0.5$.

### Spring and Slider Models

Biot’s thermodynamic formulation (Biot 1954) for a linear dissipative material when interpreted physically leads to a parallel combination of Maxwell models as shown in Fig. 6(a) (Fung 1965). Ozdemir’s model suggests that a Maxwell model can be generalized with a nonlinear dashpot, which in the limit leads to an elastoplastic model. These factors motivate us to consider models that are combinations of springs and sliders as shown in Figs. 6(b). The distributed element model of Iwan (1966) and Thyagarajan (1989) belong to this class.

### Integral Formulation of Spring and Slider Models

Zaiming and Katukura (1990) showed that the elastoplastic model of (23a) (which is nothing but the equation of a single spring-slider series combination) can be generalized in integral form to represent a multiple spring-slider model

$$M = \{U(\dot{\phi})U(M)q\}^+ + U(\dot{\phi})U(-M)f^- + U(-\dot{\phi})U(M)f^- + U(-\dot{\phi})U(-M)q^- \dot{\phi}$$

(34)
Plasticity models are smoothened for two reasons: (1) To represent distributed yielding in macroconstitutive models such as moment-curvature and force-deformation relationships; and (2) to alleviate numerical procedures near the yield point. However, it is found that such smooth models without a yield surface locally violate Drucker’s stability postulate, which states that for any load cycle with initial and final load level $M_0$, the following inequality must be satisfied: $\psi (M - M_0) \leq 0$. In Fig. 7, it can be seen that for a cycle ABC, this is not true. However, for larger cycles like CDE, the postulate holds. The current model, therefore, has limitations when $N$ does not tend to infinity. However, because this does not cause any numerical instability and such models work well for practical load histories in predicting deformations, they can certainly be used for analysis considering their other advantages. The discrete element model of Iwan (1966) has a piecewise linear transition and does not violate Drucker’s postulate. But it involves many more internal variables and does not serve the purpose of easing the numerical solution. Casciati (1989) proposed an additional hysteretic spring with negative energy dissipation, which when added to the Wen-Bouc model reduces the violation of Drucker’s postulate.

CONCLUDING REMARKS

A versatile smooth hysteretic degrading model has been formulated with rules for stiffness degradation, strength degrada-tion, and pinching. The general formulation is, however, not limited by these rules, and other rules may be used in this internal variable framework. Moreover, a general picture has been presented for representation of 1D inelastic behavior with deterioration into which different specific hysteretic models fit.

Effective solution algorithms for progressive collapse analysis of structures have to be devised that can utilize these hysteretic models. One such potential method is the state-space approach presented by Simeonov (1999), which solves the equilibrium and hysteretic constitutive equations simultaneously. Also, for more realistic nonlinear analysis, the 1D models have to be extended to 3D models featuring interaction between various stress resultants. Several attempts have been made to generalize the 1D models to three dimensions. For example, Casciati (1989) proposes a 3D Wen-Bouc type model obtained by smoothening a 3D yield surface. The relationships presented here between various models are useful in understanding and synthesizing multidimensional models. Simeonov (1999), for example, developed a smooth model including the effects of interaction of the stress resultants in a beam-column section.

The SHM and its solution algorithm have been coded as an independent program object and have been included in the computational platform IDARC2D (Valles et al. 1996).

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APPENDIX. REFERENCES
