Nonlinear Analysis of Structural Frame Systems by the State-Space Approach

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Abstract: This article presents the formulation and solution of the equations of motion for distributed parameter nonlinear structural systems in state space. The essence of the state-space approach (SSA) is to formulate the behavior of nonlinear structural elements by differential equations involving a set of variables that describe the state of each element and to solve them in time simultaneously with the global equations of motion. The global second-order differential equations of dynamic equilibrium are reduced to first-order systems by using the generalized displacements and velocities of nodal degrees of freedom as global state variables. In this framework, the existence of a global stiffness matrix and its update in nonlinear behavior, a cornerstone of the conventional analysis procedures, become unnecessary as means of representing the nodal restoring forces. The proposed formulation overcomes the limitations on the use of state-space models for both static and dynamic systems with quasi-static degrees of freedom. The differential algebraic equations (DAE) of the system are integrated by special methods that have become available in recent years. The nonlinear behavior of structural elements is formulated using a flexibility-based beam macro element with spread plasticity developed in the framework of state-space solutions. The macro-element formulation is based on force-interpolation functions and an intrinsic time constitutive macro model. The integrated system including multiple elements is assembled, and a numerical example is used to illustrate the response of a simple structure subjected to quasi-static and dynamic-type excitations. The results offer convincing evidence of the potential of performing nonlinear frame analyses using the state-space approach as an alternative to conventional methods.

1 INTRODUCTION

The formulation and solution of nonlinear inelastic structures subjected to static and dynamic loading present great difficulties and take a high toll on computational resources. This article presents a formulation and the solution of nonlinear structural analysis in state-space form applied to initial boundary value problems. The proposed solution strategy overcomes the limitations on the use of state-space models for both static and dynamic systems with quasi-static degrees of freedom. The state variables of a discrete structural system consist of global and local element quantities. The global state variables are the generalized displacements and velocities of certain nodal degrees of freedom (DOF) of the system. The local state variables are the “end” restoring forces and forces and deformations at intermediate sections, which describe the local nonlinearities. Each state variable is associated with a state equation, which characterizes its evolution in time in relation to other state variables or kinematic constraints. These are usually first-order differential or algebraic equations. Global state equations describe the motion or equilibrium of the entire system, whereas local state equations describe the evolution of the state variables needed for determining the conditions of all nonlinear elements. There is a single independent variable, represented by the “real time” in dynamic analysis and “intrinsic time” in quasi-static analysis. This article suggests a formulation of the structural elements using a flexibility approach with spread plasticity in terms of local and global state variables, which in turn assemble differential algebraic equations solved by iterative solvers.

Two alternative schemes can be considered for solving the equations of equilibrium of discrete nonlinear dynamic or quasi-static systems. First, the most commonly used solution strategy is based on implicit time-stepping meth-
methods of numerical integration; notable are the Newman, Wilson, and Humbolt methods. The response is obtained by linearization of the governing differential equations over a sequence of time steps by using the tangent stiffness and damping matrices of the system, effectively reducing the dynamic problem to a series of quasi-static ones. Iterations are necessary to reduce the error introduced by the use of the tangent matrices. Alternatively, the restoring forces in the nonlinear elements can be separated and applied as external loads on the elastic part of the system. The resulting second-order equations of motion can be reformulated as a set of first-order differential equations by adding the nodal velocities to the set of unknowns. The forces in the nonlinear elements are then defined as functions of the element deformations, deformation rates, and other variables that determine the state of the element. These differential equations are coupled with the equations of motion of the system and are solved simultaneously.

Unfortunately, this approach is applicable to structural models involving only dynamic degrees of freedom. The limitation arises from the fact that the quasi-static degrees of freedom cannot be eliminated by static condensation because the equations are formulated in the time domain and not in an incremental form in the displacement domain. This results in a system of differential algebraic equations (DAEs) instead of ordinary differential equations (ODEs). The final set of equations is a general initial boundary value problem, which must be solved by an appropriate method of integration of differential algebraic systems (DASs). The fact that ODE numerical solvers are unable to handle DAEs has prevented the generalization and widespread use of the state-space method. Only recently have reliable integration algorithms been developed and implemented in software packages specifically directed at solving DAEs, such as DASSL, RADAU5 and DASPK. These advancements have incited the DAE-based state-space method for finite-element and macro-model analysis of structural systems discussed in this article.

The state-space approach has been employed extensively in the solution of purely dynamic linear and nonlinear problems, especially in structural control and nondeterministic analysis. These problems had no quasi-static degrees of freedom and hence resulted in ODE systems with no algebraic equations. The state-space approach involving DASs has been used extensively in multibody dynamics of aerospace and mechanical assemblies. In these problems, the kinematic constraints are algebraic equations. The first application of the state-space approach to finite-element solution of quasi-static distributed plasticity problems is probably that used by Richard and Blalock to solve plane stress problems. Since this work considered only monotonic loading, the load itself (rather than time) served as the independent monotonically increasing variable. No work has been reported since that time until the beginning of this decade. DASs were used to solve large deformation plasticity problems arising in punch stretching in metal-forming operations. Papadopoulos and Taylor presented a solution algorithm based on DAEs for J2 plasticity problems with infinitesimal strain. Papadopoulos and Lu subsequently extended this strategy to a generalized framework for solution of finite plasticity problems. Iura and Atluri used the DAE-based state-space approach for the dynamic analysis of planar flexible beams with finite rotations. Fritzen and Wittekindt provided the first clear description of the methodology of formulating problems in nonlinear structural analysis at about the same time as Shi and Babuska. The work of these authors provides the basis for the approach proposed here.

This article introduces first a general procedure for identifying the state variables of a system already discretized in space by the finite-element method. The set of state variables consists of global and local (element) response quantities. Second, this article introduces the algorithm for constructing the system of state equations. The procedure accounts for the connectivity and boundary conditions of the structural model, the force-displacement relationships and the internal degrees of freedom of all elements, and the type of excitation. In general, the formulation results in a system of DAEs. Some background information on DASs is provided as well as the solution algorithm of DASSL. Third, a nonlinear bending element is formulated for modeling beam members in the framework of state-space analysis. The element falls into the category of flexibility-based continuum models, for its formulation based on force (instead of displacement) interpolation functions. Discretization across the cross section is avoided by using a constitutive macro model. Finally, the aforementioned development is implemented in a computer platform, and a numerical evaluation of the quasi-static and dynamic response of a typical civil engineering frame structure is carried out.

2 CONSTITUTIVE MODELS FOR STATE-SPACE ANALYSIS

The state-space formulation can address constitutive models based on rigorous plasticity theory as well as reasonable approximations represented by algebraic or differential relations. The existing mathematical representations of nonlinear constitutive relationships can be categorized as either time-independent or time-dependent. Models representative of the first category, such as the Ramberg-Osgood bilinear and elastoplastic models, describe the generalized stress-strain curve by algebraic equations, whereas time-dependent models employ differential equations. Although the state-space approach can easily handle the solutions based on rigorous plasticity theory, of
particular interest to this study are nonlinear constitutive representations based on the endochronic theory of plasticity. Although involving more approximations, the endochronic theory offers simpler relations described by differential equations, which can allow better understanding of the state-space approach (SSA). The intrinsic-time models make them practical alternatives to regular surface-based formulations due to (1) the absence of an explicit yield function, (2) smooth transition between the elastic and plastic states, and (3) a single equation for all phases of cyclic response. The endochronic models do not require keeping track of the yielded state; the evolutionary character of the differential equations naturally accommodates kinematic hardening. Since the constitutive equations are integrated simultaneously with the equations of equilibrium of the system, at any given time the state of the elements and the state of the system are determined exactly. The error in the solution is dictated only by the tolerance of the specific method of integration and not by the linearization of the generalized stress-strain relations.

A modified version of the endochronic model, originally proposed by Bouc-Wen and extended by Sivaselvan and Reinhorn, is used throughout this development. The total generalized restoring force $R$ is modeled as a combination of elastic and hysteretic components (Figure 1a):

$$R = [aK_0 + (1 - a)KH]u$$  \hspace{2cm} (1)

where $a$ is the ratio of postyield to elastic stiffness, $K_0$ is the initial stiffness, $KH$ is the hysteretic stiffness, and $u$ is the total generalized displacement. In this parallel-spring representation, the stiffness of the hysteretic spring is governed by

$$K_H = K_0 \left\{ 1 - \frac{R^*}{R^*_{y}} \left( \eta_1 \text{sgn}(R^*u) + \eta_2 \right) \right\} \hspace{2cm} (2)$$

where $R^* = R - aK_0u$ is the force in the hysteretic spring, $R^*_{y} = (1 - a)R_y$ is the yield force of the hysteretic spring, $n$ is a parameter controlling the transition between the elastic and plastic range, and $\eta_1$ and $\eta_2$ are parameters controlling the shape of the hysteretic loop, which must fulfill the condition $\eta_1 + \eta_2 = 1$ (according to Constantinou and Adnane). A nonlinear single-degree-of-freedom (SDOF) system subjected to dynamic and pseudo-static forces will be used to illustrate the state-space formulation (see Figure 1b). The system has three state variables and therefore three state equations. Let

$$y_1 = u \quad y_2 = \dot{u} \quad y_3 = R$$ \hspace{2cm} (3)

Then

$$m\ddot{y}_2 + c\dot{y}_1 + y_3 - F = 0$$ \hspace{2cm} (4)

Fig. 1. Schematic representation of nonlinear systems: (a) Model for restoring force; (b) inertial nonlinear SDOF system.

$$y_2 - \dot{y}_3 = 0$$ \hspace{2cm} (5)

$$\dot{y}_3 = [aK_0 + (1 - a)KH]\dot{y}_1 = 0$$ \hspace{2cm} (6)
by reordering Equation (1) and substituting the result into Equation (2):

\[ R^* = y_3 - aK_0y_1 \]  
\[ K_H = K_0 \left\{ 1 - \left| y_3 - aK_0y_1 \right|^n \right\} \]

It should be noted that the choice of state variables for this system is not unique. For example, an alternative, and probably more natural, formulation of the same problem may be devised using the hysteretic component of the restoring force as a local state variable. Let

\[ y_1 = u \quad y_2 = \dot{u} \quad y_3 = R^* \]  

Then

\[ m\ddot{y}_2 + c\dot{y}_1 + aK_0y_1 + (1 - a)y_3 - F = 0 \]  
\[ y_2 - \dot{y}_1 = 0 \]  
\[ \dot{y}_3 - K_H\dot{y}_1 = 0 \]

In this case, the hysteretic stiffness is calculated directly by Equation (2). The first version, however, is preferred. In contrast with the dynamic system in Figure 1b, a quasi-static system subjected to identical force history has only two state variables and hence two state equations. Let

\[ y_1 = u \quad y_2 = R \]  

Then

\[ y_2 - F = 0 \]
\[ \dot{y}_2 - [aK_0 + (1 - a)K_H]\dot{y}_1 = 0 \]

In this case, the constitutive equation (15) is differential, but the equation of equilibrium (14) is algebraic. Therefore, a set of differential algebraic equations must be solved to obtain the quasi-static response of a SDOF system with nonlinear restoring force. Brenan describes the solution of such equations in detail; however, for the sake of understanding, this solution is summarized below, including some simplifying assumptions.

### 3 DIFFERENTIAL ALGEBRAIC SYSTEMS (DASs)

A DAS is a coupled system of \( N \) ordinary differential and algebraic equations that can be written in the following form:

\[ \Phi(t, y, \dot{y}) = 0 \]  

where \( \Phi, y, \) and \( \dot{y} \) are \( N \)-dimensional vectors, \( t \) is the independent variable, and \( y \) and \( \dot{y} \) are the dependent variables and their derivatives with respect to \( t \).

A consistent set of initial conditions \( y(t_0) = y_0, \dot{y}(t_0) = \dot{y}_0 \) must be specified such that Equation (16) is satisfied at the initial time \( t_0 \). Since the system is coupled, some of the equations in (16) may not have a corresponding component of \( \dot{y} \). Consequently, the matrix

\[ \frac{\partial \Phi}{\partial \dot{y}} = \left[ \frac{\partial \Phi_i}{\partial \dot{y}_j} \right] \]

may be singular. A measure of the singularity is the index (defined rigorously by Brenan et al.). In a simple formulation this index is equal to the minimum number of times the implicit system in Equation (16) must be differentiated with respect to \( t \) to determine \( \dot{y} \) explicitly as functions of \( y \) and \( t \). It becomes clear that a system of ODEs written in the standard form

\[ \dot{y} = g(t, y) \]

is index 0. The system composed of Equations (4), (5), and (6), for example, can be converted to the standard form without additional differentiation:

\[ \dot{y}_2 = \frac{F - cy_2 - y_3}{m} \]
\[ \dot{y}_1 = y_2 \]
\[ \dot{y}_3 = aK_0y_2 + (1 - a)K_0 \]

\[ \times \left\{ 1 - \left| y_3 - aK_0y_1 \right|^n \right\} \]
\[ \times \left[ \eta_1 \text{sgn}(y_3y_2 - aK_0y_1y_2) + \eta_2 \right] \}

The set (14) and (15) modeling the quasi-static response of SDOF system, however, is index 1, because the algebraic Equation (14) must be differentiated once before substitution into Equation (15):

\[ \dot{y}_2 = \dot{F} \]
\[ \dot{y}_1 = \dot{F} \left[ aK_0 + (1 - a)K_0 \left\{ 1 - \left| y_2 - aK_0y_1 \right|^n \right\} \times \left[ \eta_1 \text{sgn}(y_2y_1 - aK_0y_1y_1) + \eta_2 \right] \right]^{-1} \]

This equation does not provide an explicit expression of the derivative in terms of the independent and dependent variables. However, the left- and right-hand sides are coupled only through the argument of the \( \text{signum} \) function, the sole purpose of which is to indicate load reversal. Noting that the stiffness terms in the denominator of Equation (23) are always positive, the sign of the displacement rate \( \dot{y}_1 \) follows that of the rate of the applied force \( \dot{F} \), and the \( \text{signum} \) can be evaluated from information on the latter.
The solution of DAEs is more involved than the solution of ODEs. Some of the difficulties, discussed in depth by Brenan,\textsuperscript{6} are summarized next. Since the algebraic and differential equations have different time scales, the system appears as extremely stiff, in which case implicit integration schemes have to be used. Only the differential equations require initial conditions. However, it is not possible to separate the variables associated with the differential equations from those which are not. Therefore, initial conditions are provided for all variables. These initial conditions must be consistent. Finally, DASs of index greater than 1 pose additional problems, and only a few solution algorithms can handle them. A brief summary of the integration method, implemented in DASSL, is provided for the sake of completion. This implies replacing the solution $y$ and its derivative $\dot{y}$ at the current time by a difference approximation and solving the resulting algebraic equations using a Newton-type method. The derivative is calculated by a backward differentiation formula:

$$\dot{y}_n = \frac{1}{h_n} \left( y_n - \sum_{i=1}^{k} \alpha_i y_{n-1} \right)$$ \hspace{1cm} (24)$$

where $y_n$, $\dot{y}_n$, and $y_{n-1}$ are the approximations of the solution of Equation (16) and its derivative at time $t_n$ and $t_{n-1}$, respectively, $h_n = t_n - t_{n-1}$ is the time interval, $k$ is the order of the backward differentiation formula relative to $y_n$, and $\alpha_i$ and $\beta_0$ are coefficients of the method.

Substituting Equation (24) in Equation (16) results in a system of algebraic equations:

$$\Phi \left[ t_n, y_n, \frac{1}{h_n} \beta_0 \left( y_n - \sum_{i=1}^{k} \alpha_i y_{n-1} \right) \right] = 0 \hspace{1cm} (25)$$

The iteration matrix of the Newton-type method is defined as follows:

$$N(t_n, y_n, \dot{y}_n) = \frac{\partial \Phi(t_n, y_n, \dot{y}_n)}{\partial y} + \frac{1}{h_n \beta_0} \frac{\partial \Phi(t_n, y_n, \dot{y}_n)}{\partial \dot{y}}$$ \hspace{1cm} (26)$$

The process for advancing from time $t_{n-1}$ to the current time $t_n$ is summarized by the equation

$$y_{n+1} = y_n - N^{-1} \left( t_n, y_n, \dot{y}_n \right) F \left( t_n, y_n, \dot{y}_n \right)$$ \hspace{1cm} (27)$$

where the superscript $m$ is an iteration counter.

Further discussion is beyond the scope of this article. For extensive treatment of the subject of numerical solutions of DASs and the one implemented in DASSL in particular, see Brenan.\textsuperscript{6}

\section*{4 FORMULATION OF A MULTI-DEGREE-OF-FREEDOM SYSTEM}

Let us consider the multi-degree-of-freedom (MDOF) idealization of a distributed parameter system resulting from a finite-element discretization. The governing equations of equilibrium can be written as

$$M \ddot{u}(t) + C \dot{u}(t) + R(t) = F(t)$$ \hspace{1cm} (28)$$

where $M$ is the mass matrix, $u(t)$ is the displacement vector, $C$ is the damping matrix, $R(t)$ is the restoring force vector, $F(t)$ is the forcing vector, and the dot denotes time derivative.

In the general case, the set of global state variables of the system consists of three parts: (1) generalized displacements along all free nodal degrees of freedom, (2) degrees of freedom with known nonzero displacements or displacement time histories, and (3) velocities along degrees of freedom that have masses associated with them. The displacements along constrained generalized coordinates are excluded from the solution by virtue of imposing boundary conditions on the model. Degrees of freedom with known nonzero displacements, however, must be included in the set of global state variables. This situation occurs frequently in civil engineering analysis and design, where the support motion, due to soil settlement, slope instability, or earthquakes, is often known a priori. Similar circumstances exist in displacement-controlled pseudo-static laboratory testing of structural components. Most matrix analysis or finite-element programs, research or commercial, treat the dynamic support displacement as a special analysis case and require information on the ground velocity. The method proposed here addresses this common problem by designating the known displacements as state variables. However natural this treatment is, it introduces additional algebraic equations to the set of state equations, the implications of which are discussed below.

The final segment of global state variables is comprised of the displacement rates along degrees of freedom, which have directional or rotational masses associated with them. In practical cases, the number of velocity state variables may differ substantially from the number of their displacement counterparts due to routine engineering practices of using a lumped, rather than consistent, mass matrix. Furthermore, rotational and even some directional components of mass, whose effect is presumed negligible, are often ignored. In this development, the global mass matrix is assumed diagonal. The inherent damping of the system is modeled by mass-proportional and (elastic) stiffness-proportional contributions to the global damping matrix. Hysteretic damping and damping from concentrated sources (dampers) is incorporated in the analysis by solving simultaneously the global state equations and explicit element constitutive relations, force-displacement, and force-velocity rate equations, respectively. Because of this, the damping matrix has no contributions from the individual dampers, as is customary in conventional finite-element analysis. The system-restoring force vector is assembled by
standard procedures with account of the by-node-number ordering of the global degrees of freedom and element connectivity at the nodes. The restoring forces of the individual elements, which are expressed in terms of the element state variables, are transformed to the nodal coordinate system by regular local-to-global transformations to obtain their contributions to the global vector.

The assembly of the forcing vector of the system is governed by similar considerations. External excitations acting in the form of static or dynamic forces or accelerations are defined directly in global coordinates. Distributed-element loads are substituted with equivalent nodal restoring forces. Their effect is imposed as initial conditions for the corresponding element-state variables for subsequent analysis. It must be understood that the dimension of the system mass and damping matrices, restoring force, and forcing vector is equal to the number of free nodal displacement coordinates. The constrained degrees of freedom are excluded in the assembly stage to avoid the costly reordering operations of enforcing these boundary conditions in a separate step. It also becomes clear that in the framework of the state-space approach, formation of the global tangent stiffness matrix becomes unnecessary. Equilibrium equations can be written in state-space format

\[
\mathbf{M}\ddot{\mathbf{y}}_2 + \mathbf{C}\dot{\mathbf{y}}_1 + \mathbf{R}(t) = \mathbf{F}(t)
\]  

using the transformations

\[
\mathbf{y}_1 = \mathbf{u}(t) \quad \mathbf{y}_2 = \mathbf{\dot{u}}(t)
\]

The three parts of the set of global state equations can be summarized as follows (the writing of the time dependence has been omitted for clarity):

\[
\mathbf{M}\ddot{\mathbf{y}}_{2\text{ND}} + \mathbf{C}\dot{\mathbf{y}}_{1\text{ND}} + \mathbf{R} - \mathbf{F} = \mathbf{0} \quad (31)
\]

\[
\mathbf{y}_{1\text{ND}+\text{NDH}} - \mathbf{d}_{1\text{NDH}} = \mathbf{0} \quad (32)
\]

\[
\mathbf{y}_{2\text{ND}+\text{NDH}+\text{NDV}} - \mathbf{y}_{1\text{NDV}} = \mathbf{0} \quad (33)
\]

where \( ND \) is the number of active nodal DOFs, \( NDH \) is the number of nodal DOFs with known nonzero displacements (displacement time histories), \( NV \) is the number of nodal DOFs that have directional or rotational masses associated with them, and \( NDOF \) is the total number of DOFs. Note that the mass and damping matrices in Equation (31) have been condensed from their original dimensions \((NDOF \times NDOF)\) to \((ND \times ND)\).

The state of each of the nonlinear elements of the structure must be defined by evolution equations involving the generalized restoring forces, generalized displacements, and the internal variables used in the formulation of the element model. Not all these are necessarily designated as local state variables. The idea is to define a minimal set of independent quantities and use pertinent transformations for the dependent ones. In general, the state of the element is modeled by a system of differential equations:

\[
\dot{\mathbf{R}}_e = \mathbf{G} (\mathbf{R}_e, \mathbf{u}_e, \dot{\mathbf{u}}_e, \mathbf{z}_e, \dot{\mathbf{z}}_e) \quad (34)
\]

\[
\dot{\mathbf{z}}_e = \mathbf{H} (\mathbf{z}_e, \mathbf{R}_e, \dot{\mathbf{R}}_e, \mathbf{u}_e, \dot{\mathbf{u}}_e) \quad (35)
\]

where \( \mathbf{G} \) and \( \mathbf{H} \) are nonlinear functions, \( \mathbf{R}_e \) and \( \dot{\mathbf{R}}_e \) are the generalized restoring forces and their rates, \( \mathbf{u}_e \) and \( \dot{\mathbf{u}}_e \) are the generalized displacements of the element nodes and their rates, and \( \mathbf{z}_e \) and \( \dot{\mathbf{z}}_e \) are the internal variables and their rates.

The element state equations also must establish a connection between the local state variables (generalized restoring forces) and global state variables (generalized displacements and velocities). The schematic of this interaction is presented in Figure 2.

The principle of virtual forces is used to obtain state Equation (34) for the beam element, as shown below. Determination of the tangent flexibility matrix requires information on the distribution of plastic strain along the element. This is achieved by using the strains at intermediary integration (quadrature) points as internal variables and introducing state equations tracking their evolution Equation (35).

4.1 Formulation of a flexibility-based beam macro element

The macro element is represented by equations using variables at the end of the element and at sections located at the intermediary quadrature points. The nonlinear functions \( \mathbf{G} \) and \( \mathbf{H} \) in Equations (34) and (35) are derived using both end variables and variables at quadrature sections. Direct relationships between the end force and displacement rates are only now becoming available for line elements subjected to a combination of bending, shear, and axial load.\(^{25}\) The process of defining a capable macro element requires (1) nonlinear material stress-strain relations, (2) relations representing differences between the apparent material characteristics exhibited in monotonic and cyclic

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Fig. 2. Interaction between local and global states.
loading, and (3) information on deterioration of material properties, expressed as pinching of hysteretic loops or stiffness or strength degradation. Although usually more pronounced in members made of composite materials, such as reinforced concrete, the latter occurs commonly in steel connections and soil-foundation interfaces.

In the present formulation, the nonlinear moment-curvature relationships of cross sections (slices) at discrete locations along the element axis are chosen to represent constitutive laws for bending response only. The formulation including the shear and axial effects\(^{25}\) is more complex and beyond the scope of introducing the element’s basic formulation. The convenient macroscopic formulation has found utility in modeling both material plasticity and deterioration phenomena. The effect of incorporating these factors in a predefined model of section response propagates to the element constitutive relations. To capture the effects of cyclic loading on the apparent material properties, the monotonic moment-curvature curves are assumed to represent the cyclic response of the section. Experimentally calibrated rules for the deterioration events\(^{26}\) discussed earlier could then be used to degrade the monotonic envelope. The smooth transition between the elastic and plastic states of response can be calibrated by considering the results of curvature-controlled analysis of a section of beam member discretized into a large number of fibers. In the extreme case of bilinear uniaxial stress-strain curves of the individual filaments, the resulting moment-curvature relationship has a smooth transition between the elastic and plastic stages of response due to the distributed yield of the section. The actual properties of materials such as steel, concrete, and aluminum, commonly used for constructing bending elements, are multilinear rather than bilinear, which leads to an even smoother transition. Furthermore, aluminum and some alloys exhibit nonlinearity throughout all stages of response, including the initial elastic range. The formulation in Equations (1) and (2) has the ability to model smooth degrading pinching constitutive relations, making it a natural choice for the development of the element presented next.

The state variables chosen for the beam macro element are the independent end forces and the curvatures of sections located at the quadrature points:

\[
y_{e(1:3)} = \hat{\mathbf{R}}_e \]

\[
y_{e(4:3+NG)} = \phi \]

where \(\hat{\mathbf{R}}_e = \{F_1 \ M_1 \ M_i \}^T\) are the element nodal forces, including the axial force at one end and the bending moments at both ends, and \(\phi = \{\phi_1 \ \phi_2 \ \ldots \ \phi_{NG}\}^T\) are the section curvatures at \(NG\) quadrature points.

The state equations define the constitutive laws of the element and the individual sections:

\[
\ddot{\mathbf{R}}_e = \mathbf{K}_e \ddot{\mathbf{u}}_e \]

\[
\ddot{\phi}_e = \mathbf{K}_e \ddot{\phi}_e \]

where \(\mathbf{K}_e\) is the element stiffness matrix of size \(3 \times 3\), the derivation of which is described later. \(\ddot{\mathbf{u}}_e = \{\ddot{u} \ \ddot{\theta}_1 \ \ddot{\theta}_2 \}^T\) are the axial deformation and chord rotations at the ends, \(\mathbf{M} = \{M_1 \ M_2 \ \ldots \ M_{NG}\}^T\) is the total bending moment at monitored sections at the quadrature locations, \(\mathbf{I}\) is the identity matrix of size \(NG \times NG\), \(\mathbf{a} = \text{diag}[a_1 \ a_2 \ \ldots \ a_{NG}]\) is the ratio of postyield to elastic stiffness of monitored sections, \(\mathbf{K}_0 = \text{diag}[K_{0,1} \ K_{0,2} \ \ldots \ K_{0,NG}]\) is the elastic stiffness of monitored sections, and \(\mathbf{K}_H = \text{diag}[K_{H,1} \ K_{H,2} \ \ldots \ K_{H,NG}]\) is the hysteretic stiffness (rigidity) of monitored sections.

Equation (39) expresses the decomposition of the total moment into elastic and hysteretic components. The hysteretic stiffness Equation (2) of a section is a function of both the section curvature and the hysteretic moment. The latter is obtained by subtracting the elastic component from the total moment. The moment rates at the integration points are found by interpolation of the end moment rates, yielding a direct relationship between the section curvatures and the end moments:

\[
\mathbf{b}_G \mathbf{M}_e = [\mathbf{aK}_0 + (I - a)\mathbf{K}_H] \phi
\]

where \(\mathbf{b}_G\) maps the local end degrees of freedom to the element deformations \(\mathbf{u}_e\) (Figure 3):

\[
\mathbf{b}_G = \begin{bmatrix}
\frac{x_1}{L} - 1 & \frac{x_1}{L} \\
\frac{x_2}{L} - 1 & \frac{x_2}{L} \\
\vdots & \vdots \\
\frac{x_{NG}}{L} - 1 & \frac{x_{NG}}{L}
\end{bmatrix}
\]

is the moment interpolation matrix.

The transformation \(\mathbf{T}_e\) maps the local end degrees of freedom to the element deformations \(\mathbf{u}_e\) (Figure 3):

\[
\mathbf{u}_e = \mathbf{T}_e \mathbf{u}_e
\]

\[
\mathbf{T}_e = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
0 & \frac{1}{L} & 1 & 0 & -\frac{1}{L} & 0 \\
0 & \frac{1}{L} & 0 & 0 & -\frac{1}{L} & 1
\end{bmatrix}
\]
where \( \mathbf{u}_e = \{ u_i, v_i, \theta_i, u_j, v_j, \theta_j\}^T \) are the local generalized displacement coordinates.

Combining Equations (38) and (41) renders a formulation of the element force-displacement relation suitable for considering the connection between local and global degrees of freedom:

\[
\dot{\mathbf{R}}_e = \mathbf{K}_e \mathbf{T}_e \dot{\mathbf{u}}_e
\]

(42)

Transforming the global generalized displacement coordinates of the end nodes into their local counterparts by standard global-to-local operators achieves the goal of writing Equation (42) entirely in terms of state variables. Recall that the active nodal degrees of freedom, including those with known displacement time histories, have already been designated as system state variables:

\[
\dot{\mathbf{u}}_e = \mathbf{T}_g \dot{\mathbf{u}}_g
\]

\[
\ddot{\mathbf{R}}_e = \mathbf{K}_e \mathbf{T}_e \mathbf{T}_g \dot{\mathbf{u}}_g
\]

(43)

(44)

where \( \mathbf{T}_g \) is a global-to-local transformation matrix of size \( 6 \times 6 \).

The constitutive relations at both the element and section levels have been written entirely in terms of state variables considering the fact that the stress-strain responses (Equation 40) of monitored sections are defined as functions of the end moments \( \mathbf{M}_e \) and curvatures \( \phi \). The element state equations result from Equations (44) and (40):

\[
\dot{\mathbf{y}}_{e(1:3)} - \mathbf{K}_e \mathbf{T}_e \mathbf{T}_g \dot{\mathbf{y}}_{1.e} = 0
\]

(45)

\[
\dot{\mathbf{b}}_G \dot{\mathbf{y}}_{e(2:3)} - \left[ a\mathbf{K}_0 + (1 - a)\mathbf{K}_H \right] \dot{\mathbf{y}}_{e(4:3+NG)} = 0
\]

(46)

Equation (44) establishes one of the necessary links between local and global state variables. The restoring forces of the element, assembled in the global vector \( \mathbf{R}_e \), represent its share in resisting the load imposed on the system. The full set of end forces \( \mathbf{R}_e \) is generated by equilibrium transformations of the independent forces \( \ddot{\mathbf{R}}_e \):

\[
\mathbf{R}_e = \mathbf{T}_e^T \ddot{\mathbf{R}}_e
\]

(47)

where \( \mathbf{R}_e = \{ F_i, V_i, M_i, F_j, V_j, M_j\}^T \).

The element contribution \( \mathbf{R}_e^g \) to the global restoring force vector \( \mathbf{R} \) is obtained by local-to-global transformation of the end forces:

\[
\mathbf{R}_e^g = \mathbf{T}_g^T \mathbf{T}_e^T \ddot{\mathbf{R}}_e
\]

(48)

4.2 Flexibility matrix of beam macro element

The stiffness matrix \( \mathbf{K}_e \), which links the independent end forces and displacements, is derived by inverting the element flexibility matrix. The advantage of a flexibility-based over a stiffness-based formulation is that the element force field can be established exactly from information on the end forces. The deformation field, however, cannot be described accurately given the end displacements.* The force interpolation functions, on the other hand, are exact because beam elements are internally determinate. They enforce linear variation of the bending moment rate and constant distribution of the shear and axial force rates such that the equilibrium conditions are satisfied (Figure 4):

\[
\mathbf{R}_e(x) = \mathbf{b}(x) \ddot{\mathbf{R}}_e
\]

(49)

where \( \mathbf{R}_e(x) = \{ F(x), V(x), M(x)\}^T \) are the section forces (stress resultants) and

\[
\mathbf{b}(x) = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -\frac{1}{L} & -\frac{1}{L} \\
0 & \frac{x}{L} & -1 \frac{x}{L}
\end{bmatrix}
\]

is the force interpolation matrix.

The definition of the flexibility-based beam element is based on the principle of virtual forces. The relationship between the end force and displacement rates is obtained by equating the internal and external virtual work:

\[
\delta \mathbf{R}_e^T \dot{\mathbf{u}}_e = \int_0^L \delta \mathbf{R}_e(x)^T \dot{\epsilon}(x) \, dx
\]

(50)

where \( \epsilon(x) = \{ \epsilon(x), \gamma(x), \phi(x)\}^T \) are the section deformations, and \( \delta \mathbf{R}_e \) are virtual forces.

The rates of deformation at a particular location \( x \) are related to the respective stress resultant rates through the tangent matrix of section flexibility distributions \( \mathbf{f}(x) \):

\[
\dot{\epsilon}(x) = \mathbf{f}(x) \mathbf{R}_e(x)
\]

(51)

\[
\mathbf{f}(x) = \begin{bmatrix}
\frac{1}{EA} & 0 & 0 \\
0 & \frac{1}{GJ} & 0 \\
0 & 0 & \frac{1}{[aK_0+(1-a)K_a]}
\end{bmatrix}
\]

* The shape functions needed for direct generation of the element stiffness matrix by the virtual displacement method yield correct interpolation of the displacements (and, therefore, their second-order spatial derivatives) only for the case of elastic prismatic members.
The tangent flexibility matrix $F$ of the element is derived by substituting the equilibrium condition (49) and the constitutive relation (51) in the virtual work equation (50):

$$F_e = \int_0^L b(x)^T f(x) b(x) \, dx$$

All changes of material behavior along the element axis are reflected in the matrix of flexibility distributions. The axial, shear, and bending constitutive equations (51) are coupled if a section is predicted to respond inelastically by a predefined yield criterion. The matrix of section flexibility has off-diagonal terms in the plasticized regions of the element. The axial force-bending moment interaction has been studied extensively, and numerous models, the majority of which are based on the classic theory of plasticity, have been developed over the years. The effect of the combined action of shear and moment on the respective capacities and flexibilities has received its share of attention, resulting in empirical formulas. In practical situations, however, shear-moment interaction, other than the inherent coupling by equilibrium, is rarely considered, and the shear rigidity is considered elastic until failure. It should be noted that this article aims at presenting a general algorithm for defining a flexibility-based beam element in the framework of state-space analysis. The plastic coupling of the section constitutive relations is contained in Equations (40) and (52), which express the essence of the method. Detailed treatment of axial force-bending moment interaction in two dimensions and biaxial moment interaction in three-dimensional space is beyond the scope of this article.

The Gauss-Legendre quadrature formulas are commonly used to integrate Equation (52) numerically. Unfortunately, this standard scheme does not use integration points at the ends of the interval of integration. The Lobatto quadrature rule samples the ends of the element in order to detect immediately the onset of nonlinear effects and provides a suitable solution to this problem. Its $n$ integration points, symmetrically placed with respect to the center of the interval, integrate exactly a polynomial of order $(2n-3)$. In this development, the nodes of the Lobatto quadrature control the pattern of intraelement discretization of the nonlinear bending element. The coordinates of the monitored sections directly follow the abscissas list of the rule.

5 NUMERICAL APPLICATIONS

To illustrate the method, the system of state equations of the example structure in Figure 5 is assembled and solved for two different types of excitations: (1) quasi-static and (2) dynamic. The simple portal frame was selected to illustrate explicitly the approach; however, complex structures can be handled easily. The selected loading histories induce strong inelastic response with multiple excursions in the nonlinear range. The quasi-static analysis uses displacement-controlled loading, commonly employed in laboratory testing of components. The dynamic analysis uses a ground-acceleration record from the 1994 Northridge earthquake.

The response of the state-space macro-element model is compared with finite-element solutions of the same problem using standard time-stepping algorithms in combination with Newton-Raphson iterations. The ANSYS1 analysis platform is chosen among the available programs because it hosts a nonlinear bending element representative of an alternative trend in modeling the effect of material plasticity on the element constitutive relations.

The span, height, section characteristics, and material properties of the model structure are derived from an actual design. The section constitutive model, mathematically expressed by Equations (1) and (2), requires definition of (1) the initial bending rigidity $K_0$, (2) the postyield bending rigidity $aK_0$, (3) the parameter $n$ controlling the smoothness of transition, and (4) a discrete yield point $M_y$. An assumption for the latter is clearly needed, although the margin between the yield and plastic moment of wide-flanged I-sections is relatively narrow. The bending moment, which causes full plastification of both flanges, is considered a reasonable idealization of the yield point. It is also straightforward to calculate for any level of axial load. Figure 6 presents a comparison between the moment-curvature response obtained by deformation-controlled loading of a fiber model of the column cross section and the solution of the differential equation (1) in combination with Equation (2) for the parameters in Table 1. The function of the curvature variation with time, needed for the integration, is assumed sinusoidal; however, it could be any smooth function, since time is only an auxiliary variable in the case of quasi-static loading.

In all analysis cases, the macro-element model for the proposed state-space solution consists of three bending elements. The number of internal degrees of freedom needed...
Fig. 6. Moment-curvature response of column cross sections.

Table 1
Cross-sectional properties and parameters of sections in example

<table>
<thead>
<tr>
<th>Element</th>
<th>$A$ (cm²)</th>
<th>$I$ (cm⁴)</th>
<th>$\alpha \approx E_T/E$ (%)</th>
<th>$n$</th>
<th>$M_y$ (kNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Columns</td>
<td>57.99</td>
<td>4504.98</td>
<td>3.03</td>
<td>5 or 10</td>
<td>117.05</td>
</tr>
<tr>
<td>Beam</td>
<td>73.96</td>
<td>12535.21</td>
<td>3.03</td>
<td>5 or 10</td>
<td>215.67</td>
</tr>
</tbody>
</table>

for accurate representation of the response of the corresponding member of the structure is established in the following paragraph. To simplify the examples, we have assumed (1) elastic axial force–average strain and shear force–shear angle relations, (2) no interaction between axial force and moment, and (3) infinite shear rigidity of the cross section of the element. Under these assumptions, the flexibility matrix Equation (52) can be evaluated numerically as follows:

$$ F_{1.1} = \frac{L}{EA_0} $$(53)

$$ F_{2.2} = \sum_{n=1}^{NG} \left( \frac{x_n}{L} - 1 \right)^2 \frac{1}{[a_nK_{0n} + (1 - a_n)K_{Hn}]} \omega_n \frac{L}{2} $$

$$ F_{2.3} = \sum_{n=1}^{NG} \left( \frac{x_n}{L} - 1 \right) \times \frac{x_n}{L} \frac{1}{[a_nK_{0n} + (1 - a_n)K_{Hn}]} \omega_n \frac{L}{2} $$

$$ F_{3.3} = \sum_{n=1}^{NG} \left( \frac{x_n}{L} \right)^2 \frac{1}{[a_nK_{0n} + (1 - a_n)K_{Hn}]} \omega_n \frac{L}{2} $$

where $x_n$, $a_n$, $K_{0n}$, $K_{Hn}$, and $\omega_n$ are the $x$ coordinate in the element coordinate system, the ratio of postyield to initial bending rigidity, the initial bending rigidity, the hysteretic bending rigidity, and the integration weight of a section at the $n$th Lobatto quadrature point, respectively.

The finite-element model in ANSYS¹ was created using the thin-walled plastic beam element BEAM 24 (ANSYS¹).

The latter belongs to the class of stiffness-based fiber element models. The cross sections of the frame members were divided into 10 layers parallel to the bending axis, the maximum that could be produced with the default number of points. Three strips of equal thickness were defined in each flange, two in the upper and lower eighths of the web height and two in the remainder of the web. Only the axial stresses and strains were used for determining the effect of material nonlinearity; the shear components were neglected. The constitutive relation is assumed bilinear with properties $(E, \sigma_y, E_T)$ identical to those used to obtain the moment-curvature relationship for the flexibility-based element. To have the effect of material nonlinearity accumulated through the thickness and length, the fiber moduli are first integrated over the cross-sectional area, and then the integral for the tangent stiffness matrix is evaluated along the element length. This integration uses a standard two-point Gauss rule, while the integration through the depth depends on the user definition of the cross-sectional layers. A note of interest is that the quadrature points along the length are interior, while the stress evaluation and, therefore, the calculation of the individual fiber moduli are performed at the two ends and the middle of the element. The values at the integration points are computed by assuming linear distribution of the tangent moduli between adjacent stress evaluation points. The differences between the ANSYS¹ beam model and the proposed element are summarized in Table 2.

5.1 Quasi-static displacement analysis

The node and element numbering scheme are shown in Figure 7. Table 3 maps (1) the generalized nodal displace-
ments of the structure (Figure 7) into the global state variables and (2) the independent generalized end forces and curvatures at the quadrature points of each element into the local state variables. The indices indicate position in the solution vector.

State variables \( y_{11} \) to \( y_{20}, y_{26} \) to \( y_{35}, \) and \( y_{41} \) to \( y_{50}, \) not shown in the table, represent curvatures of sections located at the interior quadrature stations. The compatibility of the displacements between the ends of connected elements is accounted for implicitly by denoting them by the same global displacement state variables, since the beam-column connections are assumed rigid. For example, at joint 2,

\[
x\text{-displacement}|_{\text{element } 1} = x\text{-displacement}|_{\text{element } 2} = u_4 = y_1 \\
y\text{-displacement}|_{\text{element } 1} = y\text{-displacement}|_{\text{element } 2} = u_5 = y_2 \\
\text{Rotation}|_{\text{element } 1} = \text{rotation}|_{\text{element } 2} = u_6 = y_3
\]

![Fig. 7. Node and element numbering and active displacement DOF.](image)

**Table 3**

<table>
<thead>
<tr>
<th>Global and local state variables for quasi-static displacement analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>7 Global</strong></td>
</tr>
<tr>
<td>State variable</td>
</tr>
<tr>
<td><strong>8 Element 1</strong></td>
</tr>
<tr>
<td>State variable</td>
</tr>
<tr>
<td><strong>9 Element 2</strong></td>
</tr>
<tr>
<td>State variable</td>
</tr>
<tr>
<td><strong>10 Element 3</strong></td>
</tr>
<tr>
<td>State variable</td>
</tr>
</tbody>
</table>

In the general case of a flexible joint, explicit compatibility state equations could be written. For example, the rotational flexibility of joint 2 could be incorporated easily using the following compatibility state equation:

\[
\text{rotation}|_{\text{element } 1} - \text{rotation}|_{\text{element } 2} = \text{rotation}|_{\text{joint } 2}(\text{joint flexibility})
\]

and introducing new functions for the resulting additional state variables. In the examples presented here, however, the beam-column connections are assumed rigid.

The equations of static equilibrium are obtained by removing the inertial and damping forces from Equation (31). The global state equations (57) involve only active DOFs and DOFs with known displacement histories.

\[
\begin{bmatrix}
y_4 \\
y_7 + (y_{23} + y_{24})/L_2 \\
y_9 + y_{23} \\
y_{22} + (y_{38} + y_{39})/L_3 \\
-(y_{23} + y_{24})/L_2 - y_{37} \\
y_{24} + y_{38}
\end{bmatrix} = \begin{bmatrix} d(t) \end{bmatrix} = 0
\]

where \( d(t) \) is the displacement history applied at node 2, and \( L_1, L_2, \) and \( L_3 \) are the lengths of the respective elements.

The state of each element is defined by a pair of equations; see Equations (58) to (63) below, in which the parenthesized superscript refers to the element number.

\[
\begin{bmatrix}
y_{10} \\
y_{11} \\
\ldots \\
y_{21}
\end{bmatrix} = \begin{bmatrix} \mathbf{r}_e \end{bmatrix} = 0
\]

\[
\begin{bmatrix}
y_{22} \\
y_{23} \\
y_{24}
\end{bmatrix} = \begin{bmatrix} \mathbf{r}_e \end{bmatrix} = 0
\]

\[
\begin{bmatrix}
y_{37} \\
y_{38} \\
y_{39}
\end{bmatrix} = \begin{bmatrix} \mathbf{r}_e \end{bmatrix} = 0
\]

\[
\begin{bmatrix}
y_{40} \\
y_{41} \\
\ldots \\
y_{51}
\end{bmatrix} = \begin{bmatrix} \mathbf{r}_e \end{bmatrix} = 0
\]

It is noteworthy that performing all matrix operations in these equations yields the exact form of the system of DAEs assembled for DASSL.
A benchmark is established using the ANSYS\textsuperscript{1} finite-element model by reducing the mesh size until any further refinement does not change the solution. This approach recognizes the fact that the error due to inaccurate interpolation of the strain field in stiffness-based nonlinear bending elements could be minimized by finer discretization.\textsuperscript{18}

The quasi-static displacement history in Figure 8 was applied at the top of column 1. After several analyses, it was established that doubling the number of elements in the columns from 40 to 80 causes no apparent changes in the (rotation) solution. The beam remains elastic and does not necessitate subdivision. The response of a model with 80 elements in each column will serve as a benchmark for verification in all analysis cases considered in this article.

A note of interest about the ANSYS\textsuperscript{1} solution is that the Newton-Raphson iterations are performed until the norm of the force imbalances falls within the tolerance limit of 0.1 percent of the norm of the applied load. A line-search technique is activated to accelerate convergence.

The macro-element model (Figure 7) for the proposed state-space solution requires only one nonlinear bending element for each frame member, in contrast to the solution by finite elements (ANSYS\textsuperscript{1}). The number of quadrature points is increased to achieve accuracy comparable with the benchmark model. This alternative method of refinement is applicable because the element model is flexibility-based and introduces only numerical integration error in the global solution. Results of analyses with gradually increasing numbers of quadrature points (not shown here for the sake of brevity) reveal a consistent trend of convergence to the benchmark solution. A comparison of quasi-static response with the densely meshed ANSYS\textsuperscript{1} model shows that 12 integration points per element yield adequate accuracy of the proposed state-space model. The same number of quadrature points will be used in all subsequent analyses.

The minor differences between the two solutions in the nonlinear range are mostly due to the different section constitutive models. As expected, the transition between the elastic and plastic phases is better approximated by the fiber model. The smooth macro model, however, provides an adequate representation with far less computational effort. The quality of the fit could be improved by adjusting the estimate for the yield moment \( M_y \) of the I-section (see Table 1) to include some web yielding. However, we feel that the fully plastic moment overestimates the yield point (as defined in the constitutive macro model) and prefer the engineering way of calculating it.

### 5.2 Dynamic acceleration analysis

The ground acceleration at the north abutment of the SR14/I5 bridge in California, which collapsed during the 1994 Northridge earthquake, was used as input for a nonlinear dynamic analysis. The inherent damping of the structure is assumed to be 5 percent of critical, modeled by a mass-proportional damping matrix. The elastic periods of vibration are estimated at 0.75 and 0.05 s. The mass of the structure is lumped at the joints of the frame. For clarity, only horizontal directional components of mass are considered. Correspondingly, the horizontal velocities of nodes 2 and 3 (see Figure 7) are added to the global state variables of the system (Table 4).

Due to the insertion of the mass degrees of freedom, two new state equations (65) are appended to the global set (64):

\[
\begin{bmatrix}
    m_1 \dot{y}_7 \\
    0 \\
    0 \\
    m_2 \dot{y}_8 \\
    0 \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    \alpha m_1 \dot{y}_1 \\
    0 \\
    0 \\
    \alpha m_2 \dot{y}_4 \\
    0 \\
    0 \\
\end{bmatrix}
+ 
\begin{bmatrix}
    -(y_{10} + y_{11})/L_1 + y_{24} \\
    -y_9 + (y_{25} + y_{26})/L_2 \\
    y_{11} + y_{25} \\
    -y_{24} + (y_{40} + y_{41})/L_3 \\
    -(y_{25} + y_{26})/L_3 - y_{39} \\
    y_{26} + y_{40} \\
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
    -m_1 \ddot{y}_g \\
    0 \\
    0 \\
    -m_2 \ddot{y}_g \\
    0 \\
    0 \\
\end{bmatrix}
= 0
\]

\[
\{ y_7 \ y_8 \}^T - \{ \dot{y}_1 \ \dot{y}_4 \}^T = 0
\]
where $m_1 = m_2 = 24.9626 \text{ kN} \cdot \text{s}/\text{m}^2$ are the lumped masses; $\alpha = 0.8378$ is the mass-proportional damping coefficient, and $\ddot{u}_b(t)$ is the base acceleration.

As expected, the local state equations (Equations 58 through 63) are identical to those for the quasi-static force case, except for the translated position indices of the state variables in the solution vector. The analysis results in Figure 9 show good agreement with the benchmark. The differences are less than 1 percent at major peaks and somewhat higher at low-stressed excursions.

6 REMARKS AND CONCLUSIONS

This article presents a general formulation for macro-element analysis in state space. The state variables of a typical discrete system are separated into global and local. The state of each of the elements of the structure is determined by evolution equations involving the end forces, deformations, deformation rates, and other internal variables that are specific to each element type. The fundamental idea of the state-space approach is to solve in time the global equations of motion (equilibrium) simultaneously with all evolution equations of all nonlinear elements. The problems, which involve also quasi-static degrees of freedom, can be resolved in state space by using an appropriate method for integration of the resulting system of differential algebraic equations.

This article presents the formulation and solution of a flexibility-based nonlinear bending macro element for structural analysis in the framework of the state-space approach (SSA). The local state variables are the independent end forces and the curvatures at intermediary integration (quadrature) sections. The formulation employs force-interpolation functions, which strictly satisfy equilibrium, regardless of the geometry. The stress-strain relations at the quadrature locations are represented by a differential constitutive macro model based on summation of the elastic and hysteretic components of the section moment. The element flexibility matrix is evaluated by numerical integration. The accuracy can be controlled by intraelement discretization, i.e., changing the number of sampling points. Unlike element subdivision, which implies increasing the number of elements, this operation involves less computational effort.

The proposed solution strategy is compared with conventional finite-element method, based on time-stepping integration algorithms in combination with Newton-Raphson iterations. The general nature of the formulation was illustrated with two numerical examples dealing with the analysis of the same typical structure to different types of excitation: quasi-static displacements and ground accelerations. These cases are solved with a computer program implementing the state-space approach and a finite-element platform (ANSYS). All results show excellent agreement and demonstrate that the state-space analysis can provide an alternative to conventional methods.

ACKNOWLEDGMENTS

We gratefully acknowledge the financial support by the Multidisciplinary Center for Earthquake Engineering Research (MCEER), which is supported by National Science Foundation, the State of New York, and the Federal Highway Administration.

REFERENCES


Fig. 9. Ground acceleration analysis: shear versus drift column 1.