INTRODUCTION TO DYNAMIC AND STATIC INELASTIC ANALYSIS TECHNIQUES

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Abstract

The current seismic design procedures imply that structures will respond with inelastic deformations during the design seismic event. An inelastic time history analysis is usually necessary to determine the structure’s response. Design techniques using elastic spectra modified for inelastic response are currently used. Such techniques cannot determine the inelastic deformations properly. This paper presents methods based on a composite acceleration and inelastic displacement spectrum, which describes the seismic demand, and inelastic capacity functions describing the structural behavior. The inelastic response in terms of accelerations and displacements can be evaluated simultaneously from the above mentioned functions, accurately, for a single-degree-of-freedom (s.d.o.f.) system. The method is extended with some approximations to multi-degrees-of-freedom (m.d.o.f.) systems. The response demands obtained from this method are used along with a composite energy displacement spectrum to determine the seismic performance of a structure by monitoring the damage levels and states. Although approximated, the procedure produces sufficiently accurate results as shown by a case study evaluated numerically herein.

Introduction

The response spectrum analysis is a well recognized method to evaluate the seismic response of structures. Traditionally the elastic response spectra are modified for the inelastic response using well-known tools (Newmark and Hall, 1982), based on post-elastic ductility. However, such a technique is directed only towards the evaluation of acceleration response and force demands. In the inelastic response of structures, the inelastic deformations are as important as the acceleration response. Therefore, the quantification of deformation and force responses are crucial to the evaluation of seismic response of inelastic structures.

The methods available to the engineer today for such evaluations are based on response history or those prescribed by seismic standards. Methods based on simplified approximations are suggested by seismic codes (UBC 1994; ATC 33.03, and suggested update of NEHRP/1997). According to these methods, the required strength of the structure is described as a fraction (1/R) of the expected elastic response, and the inelastic deformations are determined using multiples (C_d) of the elastic deformations obtained from the response of the structure with the reduced elastic forces. The rational behind the reduction factor R, related to the “desired” inelastic deformation, suggested by Krawinkler et al., (1992); Fajfar et al., (1992); Chang and Mander (1994), among others, links the inelastic deformations (or ductilities) to the initial period of the structure and type of seismic motion. Such relations were also investigated by current evaluations of seismic standards. These types of relations can be used to define inelastic response demands based on inelastic response spectrum, as outlined further in this paper. More recently a methodology based on capacity spectrum was developed (Dierlein et al, 1991.). The response is estimated using the elastic response spectrum in conjunction with an inelastic capacity curve for the structure and produces only the acceleration response. The inelastic capacity function of a structure that links the monotonic force and displacement up to collapse was used by Freeman (1994) in conjunction with elastic acceleration and displacement response spectra to determine response demands. The determination of the response quantities included the plot of acceleration spectra versus the pseudo-displacement spectra derived from the acceleration spectrum. However, the response spectra used for this purpose, were related to the elastic response only. The inelastic hysteretic effects were considered through an equivalent viscous damping coefficient and an increased period, both empirical in nature.
Composite acceleration and pseudo-displacement spectra were also used by C. Kircher (1993) and Naeim et al. (1994) to compare the effects of various earthquakes. More recently, Reinhorn et al. (1995) developed composite spectra for elastic structures and used this data to evaluate the response of structures with added damping. A similar technique is extended herein to inelastic structures characterized by their initial dynamic properties and by their inelastic strength. The paper suggests to evaluate simultaneously the inelastic deformation and acceleration demands for simple structures defined by single (s.d.o.f.) and multiple-degree-of-freedom (m.d.o.f.) models through a spectral approach. While the developments are based on the inelastic response principles, several assumptions had to be made, which impose some limitations to the application of this method to complex inelastic systems, such as m.d.o.f. structures, discussed in the paper.

### Inelastic time history analyses

A simple dynamic system will respond to a support motion \( \ddot{u}_g \) depending on its dynamic properties linked to its mass \( m \), structural restoring force \( Q(u) \) and internal structural damping \( c \) according to:

\[
m \ddot{u}(t) + c \dot{u}(t) + Q(u(t)) = -m \ddot{u}_g(t)
\]

where \( u(t) \) is the displacement response, and the overdot indicates the time derivative operand. For an elastic structure the restoring force follows a linear relation:

\[
Q(u) = k u(t)
\]

In such case, the structure is characterized by its circular frequency, \( \omega_o = \sqrt{k/m} \), and its critical damping ratio, \( \xi = c / 2m \omega_o \), and the dynamic equation (Eq. (1)) reduces to its acceleration form:

\[
\ddot{u} + 2\xi \omega_o \dot{u} + \omega_o^2 u = -\ddot{u}_g
\]

However for an inelastic structure the solution of Eq(1) can be obtained only by direct numerical integration (Newmark, Wilson or other numerical schemes). Approximated solutions can be obtained using linearized equations of the type of Eq.(3) with equivalent damping ratios \( \xi_{eq} \) and equivalent frequency \( \omega_{eq} \) (Iwan and Gates, 1979). This method although approximated yields satisfactory results for a wide range of structures. This analysis requires a good knowledge of expected site motion and a very good modeling of the structure, since the time history analysis is extremely sensitive. Rigorous, yet simple inelastic methods are required for the design office. Alternative analyses include monotonic inelastic analyses and spectral methods and are explained below.

### Capacity of inelastic structures

The capacity of a structure is the resistance displayed during the dynamic response as reaction to the inertial and damping forces (function \( Q(u) \) in Eq.(1)). This is a function which develops during the dynamic response and is implicitly calculated during the time history analysis explained above. An approximation of this function can be obtained with a good prediction of inertial forces and a monotonic inelastic analysis. The nonlinear monotonic pushover analysis, or collapse mode analysis, is a simple and efficient technique to study the strength capacity of a building under expected inertial force distributions. The pushover analysis is carried out by incrementally applying lateral loads, or displacements, to the structure. Therefore, a nonlinear analysis is carried out at each load (or displacement) increment, but the number of analysis steps are considerably less than the ones involved in a time-history analysis.

The sequence of component yielding, and the history of deformations and shears in the structure can be traced, as the lateral loads (displacements) are monotonically increased. Often the results are presented in graphs that describe the variation of the story shear capacity versus story drift, or base shear capacity versus top displacement, for a global description. Along the response curve, critical stages in the response can be identified, such as first cracking or yielding in structural elements (see Fig. 1). Furthermore, strength and service limit states, such as the failure of an element, the formation of a
collapse mechanism, etc., can be marked. Therefore, the pushover analysis describes one of the loading paths which provides the monotonic force capacity response of the structure and the associated deformations.

The capacity curve determined from a pushover analysis depends on the lateral force (displacement) distribution used to load the structure (see Fig. 2). The distribution is set to approximate the actual distribution that the structure will experience during an earthquake. Three options to calculate the loading distribution can be used: (i) force control, (ii) displacement control, and (iii) modal adaptive control. In the first option, the distribution for the lateral loads is specified, and the structure is then incrementally loaded. For the displacement control pushover, a displacement profile is specified, and is then incrementally applied to the structure. Finally, the modal adaptive pushover considers an incremental load distribution that is modified, according to the instantaneous dynamic properties, as the structure responds in the inelastic range. The three pushover options are described below.

During a force control pushover analysis the structure is subjected to an incremental distribution of lateral forces. The displacements corresponding to that incremental loads are then calculated. A ‘generalized power distribution’ is introduced to consider different variations of the story accelerations with the story elevation. This distribution is introduced to capture different modes of deformation, and the influence of higher modes, in the response, similarly with the standard code distributions. The force increment at floor “i” is calculated according to:

$$
\Delta F_i = \frac{W_i h_i^k}{\sum_{l=1}^{N} W_l h_l^k} \Delta V_b
$$

where $k$ is the parameter that controls the shape of the force distribution. The recommended value for $k$ may be calculated as a function of the fundamental period of the structure ($T$): $1.0 < k = 1.0 + (T-0.5)/2 < 2.0$. Nevertheless, any value of $k$ can be used to consider different acceleration profiles. Note that $k = 0$ produce a constant variation of the acceleration (uniform load distribution, while $k = 1$ produces a linear variation (inverted triangular distribution), and $k = 2$ yields a parabolic distribution for the story accelerations.

In the displacement controlled pushover analysis, a target displacement profile is specified and applied in equal increments to the structure. The displacement increments are determined by dividing the target displacement profile by the number of steps specified. Typically, since the deformation profile is not known before the analysis, but the lateral force distribution can be approximately estimated, force control is preferred to displacement control. However, the displacement control option can be used to determine the capacity of isolated stories.

The modal adaptive pushover analysis, is a special kind of force controlled pushover in which the story force increments are not constant. A constant distribution throughout the incremental analysis assumes no variation in the acceleration profile due to inelastic behavior. Note that often the force distributions are selected considering an elastic response, although, it is clear that when the structure enters the inelastic range, the elastic force distribution may not be applicable anymore. The ‘modal adaptive distribution’ was introduced to capture the changes in the distribution of the lateral forces. Instead of a polynomial, the mode-shapes of the structure are used. Since the inelastic response of the structure changes the stiffness matrix, the mode shapes are also affected, and a distribution proportional to the changing mode shapes can capture this variation. If only the fundamental mode is considered, the increment in the force distribution is calculated according to:

$$
\Delta F_i = \frac{W_i \phi_{1i}(u)}{\sum_{l=1}^{N} W_l \phi_{li}(u)} \Delta V_b - F_i^{old}
$$

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where $\phi_{i,j}(u)$ is the value of the first mode shape at story “$i$”, variable with the extent of deformation; $V_b$ is the new base shear of the structure; and $F_{i,old}$ is the total force at floor “$i$” in the previous loading step, (see also Bracci, Kunnath and Reinhorn, 1996).

The ‘modal adaptive distribution’ can be extended to consider the contributions from more than one mode. In this case, the mode shapes are combined using the SRSS method (square root of the sum of the squares), and scaled according to their modal participation factor. The incremental force at story “$i$” is calculated according to:

$$ \Delta F_i = \frac{W_i \left( \sum_{j=1}^{Nm} \left( \phi_{i,j}(u) \Gamma_j(u) \right)^2 \right)^{1/2}}{\sum_{j=1}^{N} \left( \sum_{j=1}^{Nm} \phi_{i,j}(u) \Gamma_j(u) \right)^2} V_b - F_{i,old} $$  \hspace{1cm} (6)

where $\phi_{i,j}(u)$ is the value of the mode shape “$j$” at story “$i$”, variable with the deformation $u$; $\Gamma_j(u)$ is the modal participation factor for same mode “$j$” also variable with the deformation; $V_b$ is the new base shear in the structure; and $F_{i,old}$ is the force at floor “$i$” in the previous loading step. These can be simplified as shown below.

The ultimate capacity of a story, or building, is a key quantity for damage quantification. Using a pushover analysis, critical states in the response can be identified. However, some criteria must be used to determine what point along the capacity curve can be considered as the ultimate capacity of the structure. The following is a list, in order of increased damage to the structure, of response states that could be used to define the ultimate strength capacity of a structure at:

(a) Failure of first element
(b) Failure of first vertical gravity load carrying element - column or shear wall
(c) Formation of a full collapse mechanism

The first column or shear wall failure, option (b), is recommended to define the ultimate strength capacity of the building. Option (b) is preferred over option (a) since the loss of a beam element is not as critical as the loss of a vertical element on which other structural elements are supported. The first column or shear wall failure is preferred over the formation of a collapse mechanism, option (c), because the failure of a column or shear wall already poses a serious threat to the stability of the whole system. Note that if the collapse mechanism occurs before the column or shear wall failure, then the point along the capacity curve, corresponding to option (c), should be used as the ultimate capacity.

Besides the strength ultimate limit state, one must consider the ultimate service limit state. Often this limit is related to maximum inter-story drift ratios to avoid major damage to non-structural elements. In this report, a maximum inter-story drift of 1.2%, of the story height, is used to characterize the ultimate service capacity. Values of drift between 0.7% to 0.75% are often considered as maximum service limits, for structures with rigid partitions and glass claddings.

The capacity curves can be approximated by a set of bilinear curves (see Figs. 3) for an even more simplified analysis according to:

$$ Q(u) = Q_y \left\{ \frac{u}{u_y} - (1 - \alpha) \left( \frac{u}{u_y} - 1 \right) U \left[ \frac{u}{u_y} - 1 \right] \right\} $$  \hspace{1cm} (7)

in which, $Q_y$ and $u_y$ are the yielding strength and displacement, respectively, $\alpha$ is the post yield (hardening) stiffness ratio in respect to the initial stiffness; while $U[\cdot]$ is a step function [equals 0 for $u/u_y < 1$ or equals 1 for $u/u_y > 1$]. The same relation can be described in terms of the ductility ratio, $\mu = u / u_y$ as:

$$ Q(\mu) = Q_y \left\{ \mu - (1 - \alpha)(\mu - 1) U[\mu - 1] \right\} $$  \hspace{1cm} (8)
The bilinear capacity curves are determined considering the same post-yielding stiffness, and equal energy to failure (see Fig. 4). Equating the monotonic energy to failure:

$$A_m = \frac{1}{2} Q_y \delta_y + \frac{1}{2} (Q_u + Q_y) (\delta_u - \delta_y)$$

(9)

where $A_m$ is the work done to monotonic failure; and $Q_y$, $\delta_y$ and $\delta_u$ are the yield force, the yield deformation, and the ultimate deformation of the equivalent bilinear capacity curve. The yield force can be determined in terms of the yield displacement $\delta_y$, and the post-yielding stiffness $K_y (= \alpha K_0)$:

$$\delta_y = \frac{2Q_u \delta_u - K_y \delta_u^2 - 2A_m}{Q_u - K_y \delta_u}$$

(10)

The initial frequency $\omega_o$, (or period, $T_o$), the yielding level, $Q_y$, and the post-yielding hardening coefficient, $\alpha$, are the important parameters that describe the inelastic behavior, along with the ultimate deformation capacity ($u_u$).

Most structures (buildings, bridges, etc.) have masses distributed throughout the system. Simplified modeling of such structures considers lumped mass systems as representatives to the more complex ones. Structural systems composed of multiple components with inelastic characteristics can also be represented by a force-displacement relation or by a compatible acceleration-displacement response. Assume, therefore, a simple inelastic structure represented by a multiple-degrees-of-freedom (m.d.o.f.) system. The base shear and the displacement at the top of the structure can be considered as representative to the overall behavior.

If the structure is assumed elastic with well-defined mode shapes $\phi_j$, (mass normalized, such that $\phi_j^T M \phi_j = 1$, where T represents transposition and $M$ is the mass matrix), the modal base shear (BS), and displacement response can be represented in terms of spectral response $S(\omega, \xi)$ defined in detail in the next section:

$$BS_j = Q_{bj} = \Gamma_j S_a (\omega_j \xi_j) \text{ and } u_{bj} = \phi_{bj} \Gamma_j S_a (\omega_j \xi_j)$$

(11)

where, $\Gamma_j = \phi_j^T M r$ is the modal participation factor and $r$ is a vector of units ($r^T = \{1,1,...1\}$). If the response is dominated by the first mode, ($j=1$), a single mode can be used to completely define the response. The inertia force distribution during the dynamic response is proportional to the mode shape such that the forces at any d.o.f. (i) is:

$$F_i = m_i \phi_{ij} \Gamma_j S_a (\omega_j \xi_j) \text{ or } F_i = m_i \phi_{ij} (BS_j / \Gamma_j)$$

(12)

To obtain the force-displacement capacity relation, $Q(u)$, a monotonic inelastic analysis should be performed using incrementally increasing base shear, $BS_c$, and a force distribution, $F_u$, as described by Eq. (12) or simply using Eq. (5). Simultaneously, the modal characteristics $\Gamma_j$ and $\phi_{ij}$ should be adjusted during the analysis as the system yields and the restoring and dynamic properties change. The updated dynamic properties can be obtained through appropriate eigenvalue analyses. This type of incremental monotonic analysis with adjustment of loads is defined here-in as the “adaptable push-over” technique, (see also Bracci, Kunnath and Reinhorn, 1996).

The capacity including multiple modes can be obtained using the modal response for a set of elastic properties, $\omega_j$ and $\phi_j$, and using a square root of the sum of squares (ssrs) superposition or an equivalent more advanced combination (i.e., c.q.c., etc.). For an elastic system, or for instantaneous deformation in an inelastic system, the force (base shear, BS) and displacement responses are:

$$Q = BS = \Gamma_1^2 S_a (\omega_1 \xi_1) \cdot \text{ssrs} (\gamma_j S_{aj}) \text{ and } u_i = \phi_{ii} \Gamma_1 S_a (\omega_1 \xi_1) \cdot \text{ssrs} (\gamma_j f_j S_{dj})$$

(13)

where, the square root of sum squares superposition is used for simplicity and defined by the operand _ssrs_.

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srss \( (x_j) = \left( \sum x_j^2 \right)^{1/2} \) where, \( M \) is the number of modes considered in the analysis and the spectral modal ratios for various modes can be defined as: 

\[
s_{aj} = \frac{S_a(\omega_j, \xi_j)}{S_a(\omega_1, \xi_1)}; \quad s_{dj} = \frac{S_d(\omega_j, \xi_j)}{S_d(\omega_1, \xi_1)};
\]

\( \gamma_j = \frac{\Gamma_j}{\Gamma_1} \) and \( f_{ij} = \phi_j^q / \phi_{i1} \). The ratios \( s_{aj} \) and \( s_{dj} \) are dependent on the modal frequencies and can be approximated for simplicity by various functions. A suggested approximation is given by Valles, Reinhorn et al. (1996b). The response values in Eq. (13) are similar to the single mode contribution, i.e. Eq. (11), with an additional correction coefficient given by the \( srss \) terms.

The nodal inertial forces are given similarly by:

\[
F_i = m_i \phi_j^q \Gamma_i S_a(\omega_1, \xi_1) \cdot \text{srss} \left( f_{ij}^q \gamma_j s_{aj} \right) \quad \text{or} \quad F_i = BS \frac{m_i \phi_{ij}}{\Gamma_1} \cdot \text{srss} \left( f_{ij}^q \gamma_j s_{aj} \right)
\]

The force distribution among the degrees of freedom is similar to the one for the single mode, as shown in the previous section, with the correction factor indicated by the \( srss \) functions in Eq. (14). The capacity of the inelastic structural system, \( Q(u) \), can be determined using incremental force monotonic analysis (push-over) as previously described using force distribution defined by Eq. (14) or simply Eq. (6). Since the structure is inelastic, at each incremental load after the yielding occurred, the modal characteristics \( \phi_j \) and \( \omega_j \) change and, therefore, all terms in Eq. (14) change as a function of displacements.

All the above options of analysis are implemented in a computer platform IDARC2D Ver4.0 (Valles, Reinhorn et al., 1996a). However, all current analysis programs cannot model the loss of structural elements during analysis, therefore, all analysis results beyond the first element failure may not be accurate. Research is currently under way, at the University at Buffalo, to incorporate the loss of structural elements during analysis (Ozer, 1996). This analysis method will later be implemented in the computer program IDARC2D Ver4.1.

**Definition of seismic demand by composite response spectra**

The seismic demand can be estimated by spectral methods, however, it is dependent on the capacity curves. Attempts are done to prepare such spectra independently, to be used in simplified procedures. The spectral response is based on the maximum response of a family of single-degree-of-freedom (s.d.o.f.) systems to a well defined ground motion \( \ddot{u}_g(t) \) which is the well known response spectrum \( S(\omega, \xi_o) \) function.

The response spectrum can be obtained for displacements, \( S_d(\omega_o, \xi_o) = \max |\ddot{u}| \), for absolute accelerations, \( S_a(\omega_o, \xi_o) = \max |\ddot{u} + \dddot{u}_g| \), or for any other desired response quantities. For practical purposes the absolute acceleration response is used to determine the force acting on the support system:

\[
F(\omega_o, \xi_o) = m S_a(\omega_o, \xi_o)
\]

A typical set of response spectra are shown in Fig. 5(a) and (b).

A combination of the response spectra for the same system characteristics \( (\omega_o, \xi_o) \) will form a new function defined here as the “composite spectrum,” as shown in Fig. 5(c). For harmonic excitations, the acceleration spectrum and the displacement spectrum are approximately correlated:

\[
S_a(\omega_o, \xi_o) \cong \omega_o^2 S_d(\omega_o, \xi_o)
\]

This relation is preserved, only approximately, for random ground excitations. Therefore, the square of the frequency, \( \omega_o^2 \), can be viewed as the slope of a line crossing the composite spectral function at point \( S^0(\omega_o, \xi_o) \) (see Fig. 5(c)). For a s.d.o.f. system the composite spectra provide simultaneous
information on response displacement, $S_d$, and response acceleration, $S_a$. With a simple dimensional transformation the composite spectrum may be adjusted using Eq. (15) to provide a direct relation between displacement and force spectra (see Fig. 6(a)). Therefore, for an elastic linear system one can find the elastic response at the intersection of the elastic spectrum with the line sloped according to the natural period ($\omega_o = 2\pi / T_o$) of the structure.

**Rigorous inelastic spectra** The equation of motion, Eq. (1) can be solved using time history analysis to obtain the maximum deformation for a variety of structures having an initial frequency of $\omega_o$, a yield force resistance, $Q$, and a post yielding stiffness ratio of $\alpha$. The maximum inelastic deformation, $u_m$, and the maximum force, $F_m^I$, will define the inelastic displacement and force spectra as a function of the yielding force, $Q_y$, and the initial frequency, $\omega_o$, (or the yield displacement $u_y = Q_y / \omega_o^2 m$). The inelastic spectra of interest can be derived for selected values of the yielding strength, $Q_y$, derived from the elastic force response divided by a reduction factor, $R\mu$:

$$Q_y = S_F^E (\omega_o / \xi) / R\mu = F_m^E / R\mu$$

where $R\mu$ is a constant coefficient defined as a “strength response reduction factor,” and $F_m^E$ (or $S_F^E$), is the elastic force response. In such case, it is possible to derive the response spectra for all oscillators with same ratio between the elastic response and their yield level $(R\mu = F_m^E / Q_y)$ and an initial frequency of $\omega_o$. A typical composite inelastic response spectra is shown in Fig. 6(b) along with a generalized monotonic capacity envelope of structure (defined further in the paper).

It should be noted that the composite spectra so obtained are functions of the initial frequency, $\omega_o$, (or period $T_o$) and of the “reduction” factor, $R\mu$, obtained from the elastic response, only. A typical family of inelastic composite response spectra are shown in Fig. 7(a) linking the force response with the displacement spectra.

**Approximated inelastic spectra** The spectral curves described above are result of rigorous inelastic time history analyses based on an assumed hysteretic model (bilinear), and known accelerograms. However, for the practicing engineer a simplified inelastic spectral representation which does not require an initial time history analysis is desired. Approximations of composite spectra can be obtained from an extensive statistical analysis of time history analyses (in progress). However, some approximated curves can be derived from currently available deformation relations developed by Krawinkler and Nasser (1992), Vidic et al. (1994), and Chang and Mander (1994) of the type:

$$u_m = u_y \left\{1 + \frac{1}{c_1} (R\mu - 1)\right\}^{1/c_3}$$

where $c_1$, $c_2$, and $c_3$ are constants dependent on the ground motion frequency content defined by the corner period, $T_g$, the type of hysteretic rules, and other factors. For this paper the relation suggested by Krawinkler and Nasser (1992) for which $c_1 = c_2 = c$; $c_3 = 1$, is adopted, in which:

$$c = \frac{T_o^a}{1 + T_o^a + \frac{b}{T_o}}$$

where, $a$ and $b$ are factors defined in Appendix I. These factors consider the effects of post-yielding stiffness on inelastic response. Eq. (18) can be used to define a relation between an approximated inelastic displacement spectrum and a given elastic displacement spectrum as follows:

$$S_d^I = \frac{S_d^E}{R\mu} \left\{1 + \frac{1}{c} (R\mu - 1)\right\} \geq \frac{S_d^E}{R\mu}$$

where, the reduction factor is defined as:
The approximated inelastic acceleration spectrum can also be defined in respect to the elastic acceleration spectrum as:
\[
S_a^I = \frac{S_a^E}{R_\mu} \left[ 1 + \alpha \left( \frac{S_a^I}{u_y} - 1 \right) \right]
\]

where, \(u_y = \frac{S}{R_d}E\mu\)

The approximated composite inelastic spectra derived from Eqs. (20) and (22) can link the elastic composite spectra \(S^E_d = S^E_a\) with the reduction factor \(R_\mu\), and the function of the structure strength, \(Q_s\), with the post-elastic hardening coefficient \(\alpha\). Composite spectra obtained from Eq. (20) and (22) is shown in Fig. 7(b). It should be noted that the reduction factor, \(R_\mu\), in Eqs.(20) through (22) is dependent on the elastic acceleration response and is not a constant for all structures (see Eq.(21))

It should be noted that the above approximated and simplified composite spectra can be used as substitute for the ones generated rigorously by time history analyses. However, only the rigorous inelastic spectra are used further in this paper.

**Spectral response of an inelastic system**

The displacement and force response of an inelastic system for which the capacity diagram, \(Q(u)\), is described by a bilinear model, i.e., by the initial natural frequency \(\omega_0\) (or the initial stiffness \(k_o\)) along with the yielding level \(Q_y\) and the post yielding stiffness ratio, \(\alpha\), can be evaluated following three steps (see Fig 8):

(a) The “elastic force response”, \(F^E_m\), is determined first from the rigorous elastic composite spectra for the initial properties;

(b) A “reduction factor”, \(R_\mu\), is calculated from the ratio \(F^E_m/Q_y\). Then the structure’s inelastic composite spectra is derived by interpolation for the same \(R_\mu\);

(c) The inelastic response \((F_{d}, Q_{d}, u_{d})\) is found at the intersection of the capacity diagram, \(Q(u)\), and the rigorous (or the approximated) composite spectra curve for \(R = R_\mu\).

The response obtained as outlined above indicates the seismic force (or acceleration) demand and the seismic displacement demand, assuming that the ultimate deformation capacity, \(u_u\), at which failure occurs, is larger than this demand. If the ultimate deformation capacity is smaller than the demand, the response will lead to failure of the structure before reaching \(u_u\). The demand is influenced by the cyclic response that leads to deterioration of both strength and deformation capacity. In such case, the seismic response demand can be obtained using a deteriorated capacity curve instead of the initial capacity.

For an m.d.o.f. system with the capacity diagram based on single-mode contribution (for example \(j = 1\) in Eq.(11)), the inelastic response can be obtained similarly as for the single-degree-of-freedom (s.d.o.f.) system using the inelastic composite response spectra as outlined above. However, in such cases the relation between the forces-deformations or accelerations-displacements are not directly compatible as shown in Eq. (11). In order to obtain a compatible relation between capacity and the composite spectra, it is necessary to adjust either (i) the composite spectra or (ii) the capacity diagrams. The first option was outlined by Reinhorn et al., (1995), and the second option is suggested below. The capacity relation defined by \(Q\) and \(u\) in Eq. (11) can be adjusted by the transformation:
\[
Q_j^* = \frac{Q_j}{\Gamma_j^2} = S_a^I(\omega_j \xi_j) \quad \text{or} \quad \left( \frac{Q_j}{W} / \left( \frac{\Gamma_j^2}{g} \right) \right) = S_a^I(\omega_j \xi_j) / g
\]
The adjusted relation, \( Q^*(u^*) \), is of the same type and is compatible with the composite spectra \((S_a^2 - S_d^2)\). Using the adjusted capacity relations, the response demand is obtained by determining first the “elastic” response, \( F_m^E = S_a^E \), from the intersection with the elastic composite spectra, and then the response, \( F_d^* \) and \( u_d^* \), from intersection of capacity and inelastic composite demand curves, as shown in Fig. 8. The physical response demands, \( F_{sd} \), are then obtained by an inverse application of Eq. (23).

For multiple mode contributions the modified capacity defined previously as \( Q^* \) and \( u' \) in Eq. (10a) becomes:

\[
Q^* = \frac{Q}{\Gamma_1} \cdot \text{srss} (\gamma_j^2 s_{ij}) \quad \text{and} \quad u_i^* = \frac{u_i}{\phi_i \Gamma_1} \cdot \text{srss} (f_{ij}^y / s_{ij}) \quad (24)
\]

The inelastic demand can be obtained similarly to the single mode response, with the necessary adjustments provided by Eq. (24). It should be noted that the forcing function for the calculation of the capacity of the structure is dependent on the spectral values at each step of the analysis. However, only the first mode characteristics and spectral ratios seem to be important. Therefore, an approximation of these spectral ratios can be made successfully using a building code approach, as indicated in Appendix II, simplifying the computations. Moreover, numerical studies of regular buildings indicate that the single mode approach based on the first mode produces results almost identical to the multi-mode approach.

For inelastic structures with uniform characteristics (stiffness and strength), i.e., “regular buildings” per NEHRP and UBC recommendations, the modal shapes \( \phi_j \) do not change substantially with the loading increments. In such case, a constant force distribution (not adaptable) can be used to determine the desired capacity diagrams. This is the current procedure proposed by the ATC33.03 studies (1995). This is also the recommended approach based on this work. However, a sensitivity analysis seem to be necessary in order to determine the limitations of this approach (in progress).

In summary, the capacity diagram of a structure, independently of the accuracy with which it was obtained, can be used to obtain an estimate of the seismic demand when used with the inelastic composite spectra. For practical purposes this suggested method can use predetermined or standardized inelastic composite spectra and a simplified capacity diagram to obtain an approximation of the inelastic response without executing a dynamic time history analysis.

A building used for this case study was modeled after a structure in southern California which was retrofitted using viscoelastic dampers (Lysiak, 1996). The material properties and basic dimensions were retained, however, the building layout was simplified for this study. The building is a three-story reinforced concrete moment resisting frame structure with shear walls. The overall footprint of the building is 54.9m x 36.6m consisting of nine 6.1m bays in the longitudinal direction and six 6.1m bays in the transverse direction. The overall height of the structure is 12.2m, with the story heights all equal to 3.6m. Typical elevation and plan views are presented in Fig.9 showing shear wall and damper locations (the dampers are integrated into the chevron braces). Typical dimensions of structural elements are given in Table 1, and a more detailed description of their capacities may be found elsewhere (Lysiak, 1996).

<table>
<thead>
<tr>
<th>Story #</th>
<th>Slab Thickness (mm)</th>
<th>Column Diameter (mm)</th>
<th>Beam Width x Height (mm x mm)</th>
<th>Shear Wall Thickness x Height (mm x mm)</th>
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</thead>
<tbody>
<tr>
<td>(1)</td>
<td>155</td>
<td>460</td>
<td>205 x 610</td>
<td>205 x 2390</td>
</tr>
</tbody>
</table>

Table 1 - Typical Dimensions of Structural Elements.
The building was modeled and analyzed using IDARC-2D Ver. 4.0, which both monotonic capacity analysis (pushover) and time history analysis. The “pushover” capacity analysis provides the force-deformation envelope. The composite spectra for a family of simulated ground motions compatible to a prescribed site spectra of Southern California was calculated for various $R_\mu$ factors. The capacity and the composite spectra are used to determine graphically the inelastic response, as shown in Fig. 10.

The capacity is first adjusted to the SDOF equivalent according to Eq. 23. The initial shape and the yield level is determined using the secant approach ($Q_y^*/W = 0.27$) and the elastic response is obtained from the graph ($F_E^*/W = 0.94$). The force ratio $R_\mu$ is determined as $R_\mu = 0.94/0.27 = 3.5$. The SDOF equivalent response is obtained by interpolation as $Q^*/W = 0.30; u_d^*/H = 0.0075 (~0.75\%)$. The actual response is obtained by inverse application of Eq. 23 and is estimated as $Q/W = 0.24, u/H = 0.010 (~1.0\%)$.

The response obtained from the time history analysis for the same ground motion is used as a verification of the simplified procedure (see Fig 10). The time history response indicates $Q/W = 0.268$ and the drift ratio 0.0112 (1.12%). Therefore, the inelastic spectral procedure produces sufficiently accurate results.

The response of damped inelastic structures, in which the damping is generated in concentrated locations such as damping devices, or in well-defined structural sources, will be modified due to the changes of the composite spectrum. If the damping can be represented by a pure viscous model, then the capacity diagram will not change (Reinhorn et al., 1995) and the response can be determined from the modified composite spectrum, as shown in Fig. 11. The adjusted composite spectra are determined first for the increased damping, $S_\mu(R_\mu, \xi_o + \Delta \xi)$; then is reduced by the corresponding reduction factor $R_\mu$ as previously described to obtain the inelastic spectra $S_1(R, \xi_o + \Delta \xi)$ - see Fig 11(b). This is intersected with the modified capacity diagram to produce the inelastic response quantities. It should be noted that if only damping is added to the structure (such as with fluid type devices (Reinhorn et al., 1995)), the capacity diagram remains unchanged after retrofit as in Fig. 11(a). However, where the source of damping provides also additional stiffness, as described by Kelvin or Maxwell models (Reinhorn et al., 1995), the capacity diagram $Q(u)$, will be also modified due to the increase of stiffness (see Fig. 11(b)). The response can be obtained for the inelastic composite spectra for the enhanced viscous contribution, i.e. $S_1(R, \xi_o + \Delta \xi)$ in Fig. 11(b), intersected with a the modified capacity diagram adjusted for increased stiffness (see Fig 11(b)). The damping addition reduces simultaneously forces for small $R_o$ and $\Delta \xi$ only and displacements. An increase in stiffness further reduces deformations, but may increase overall forces (Reinhorn, 1995).

**Remarks and conclusions**

This paper presents methods to determine building response using inelastic time history analyses or inelastic spectra evaluated for various strength reduction factors from a selected ground motion or a given elastic spectrum. (Such a spectrum may be site-specific on one developed by averaging spectral demands for various ground motions). The maximum displacement and force responses are simultaneously obtained depending on the capacity envelope used to describe the inelastic characteristics of the structure. The comparison of time history analysis results and those of the simplified spectral method indicates good agreement for both single and multi-degree-of-freedom structures.

<table>
<thead>
<tr>
<th>2</th>
<th>230</th>
<th>660</th>
<th>330 x 610</th>
<th>205 x 2415</th>
</tr>
</thead>
<tbody>
<tr>
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<td>255</td>
<td>760</td>
<td>405 x 610</td>
<td>205 x 2465</td>
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</tbody>
</table>
References


Appendix I - Coefficients for Approximated Inelastic Spectra

The coefficients determined by Krawinkler and Nasser (1992) are given in Table below:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>1.0</td>
<td>0.42</td>
</tr>
<tr>
<td>2%</td>
<td>1.0</td>
<td>0.37</td>
</tr>
<tr>
<td>10%</td>
<td>0.80</td>
<td>0.29</td>
</tr>
</tbody>
</table>

$\alpha$ - postyielding stiffness (hardening) ratio

Appendix II - Spectral Ratios

The spectral ratios can be approximated by:

$$s_{aj1} = \begin{cases} 
1 & \text{for } T_i \leq T_s \\
T_i^{r} / c & \text{for } T_i > T_s \text{ and } T_j < T_s \\
(T_i / T_j)^{r} & \text{for } T_j \geq T_s 
\end{cases}$$

where $T_s$ is the cut-off period for velocity dependent range and, $r$ and $c$ the spectral constants, ($r=2/3$ in NEHRP (1995), or $r=1$ in UBC (1995) and NEHRP (1997), and $c = A_a$ in all standards).

$$s_{aj1} = \begin{cases} 
(T_i / T_j)^2 & \text{for } T_i \leq T_s \\
(T_i / T_j)^2 * T_i^{r} / c & \text{for } T_i > T_s \text{ and } T_j < T_s \\
(T_i / T_j)^{2+r} & \text{for } T_j \geq T_s 
\end{cases}$$

Similar relations, depending on periods only, can be derived based on other code spectral relations.
Figure 1: Critical stages in the Strength Capacity Diagram for a Typical Building

Figure 2: Monotonic Loading on m.d.o.f System
Figure 3: Capacity Diagram for Typical R/C Moment Resisting Frame Structure

Figure 4: Criteria used to determine the Equivalent Bilinear Capacity Curve
Figure 5: Composite Spectrum for Acceleration and Displacement

Figure 6: Elastic and Inelastic Composite Response Spectra
Figure 7: Elastic and Inelastic Response Spectra for Taft Earthquake; (a) from Time History; (b) from Approximation

Figure 8: Estimate of maximum Inelastic Response
Figure 9: Typical Plan and Elevation Views of Case Study Structure

Figure 10: Graphical Evaluation of Inelastic Response (Composite Spectra and Capacity Diagram)
Figure 11: Comparison of Structural Response before and after Retrofit