DYNAMIC TORSIONAL COUPLING
IN ASYMMETRIC BUILDING STRUCTURES

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also recognized in these studies (2, 15, 18). Techniques to overcome these difficulties by means of two-degree-of-freedom response spectra were proposed by Penzien (13, 14).

The dynamic behavior of asymmetric multistory buildings has not been as thoroughly studied. Bustamente and Rosenblueth (2) analyzed a large number of such buildings in which static eccentricities were assigned to some or all floors. On the basis of their study they concluded that a rough estimate of torsional effects could be obtained from the response of single story structures with similar characteristics. Skinner et al (18) suggested that the dynamic properties of asymmetric multistory buildings could be derived from those of similar but symmetric buildings and those of single story asymmetric structures, provided certain requirements were satisfied. These requirements consist of geometrical similarity of all floor plans, centers of stiffness (elastic centers) and of gravity located on two vertical axes, constant values of radii of inertia and stiffness, plus certain other limitations on the structural type. The two axes requirement is also given by Penzien (15), apparently without any further qualifications.

A technique whereby the dynamic properties of asymmetric multistory structures can be derived from those of their symmetric counterparts and those of two-degree-of-freedom (2-DOF) systems, is likely to reduce the computational effort appreciably. Moreover, if the natural frequencies of the 2-DOF systems are close, then the results may be used directly as input for the 2-DOF response spectrum analysis developed by Penzien (13, 14) or for the amplification curves given by Rosenblueth and Rioddy (12, 15). The purpose of the present paper is to present a general approach to the study of torsionally coupled building structures by using the properties of their uncoupled counterparts, thereby extending the scope of the methods used by Rosenblueth and Rioddy (15) and by Penzien (13, 14) to structures which may not have an elastic center. This approach is also a generalization of the techniques used in an earlier paper (17) for evaluating the natural frequencies and mode shapes of uniform
asymmetric wall-frame structures. The details developed herein are for a floor plan with one axis of symmetry so that torsional coupling is confined to one axis only, but the method as presented is also applicable to the more general case with no axis of symmetry. An exact solution satisfying the uncoupling requirements is first given for the special case in which the lateral and torsional stiffness matrices are uncoupled by the same transformation. The method is then applied to a wider class of structures where this condition is only approximately satisfied. Graphical presentations of the dynamic magnification of static eccentricity for certain structural types are given in the paper and permit a rapid evaluation of the seismic response. The paper closes with two numerical examples illustrating the use of the proposed method checking on its accuracy and comparing its results with seismic code provisions. It is believed that the proposed approach will lead to a better understanding of the nature of the lateral torsional coupling effect and hopefully will contribute to a more rational treatment of dynamic amplification in earthquake code provisions.

**ANALYTICAL MODEL**

The representation of torsional coupling in asymmetric structures by means of 2-DOF systems is carried out for two basic models. The first is a building in which the mass centers of the floors are located on one vertical axis throughout the height (point NC in Fig. 1) and the elastic centers are located on another vertical axis (point KE in Fig. 1). The second model is a building with two types of framing systems. Each system comprises several vertical planar assemblages having similar stiffness properties (e.g., frames or flexural walls) and a common variation thereof along the height. The elastic centers of these systems and the mass centers are located on three different vertical axes (points KE, SC and NC respectively, in Fig. 2). It is apparent that no single center of rigidity can be defined for this model. These two models represent, at least approximately, a wide class of high-rise building structures.
Considering the first model, the dynamic properties of the vibrating system are obtained from the solution of the free vibration equation

\[(K - \omega^2 M) \ddot{\phi} = 0\]  
(1)

or

\[
\begin{bmatrix}
K_L & 0 \\
0 & K_R
\end{bmatrix}
\begin{bmatrix}
M & M_{es} \\
M_{es} & M(e^2 + i^2)
\end{bmatrix}
\begin{bmatrix}
\ddot{\phi}_L \\
\ddot{\phi}_R
\end{bmatrix}
= \begin{bmatrix}
\ddot{c} \\
0
\end{bmatrix}
\]  
(1a)

in which \(K_L\) and \(K_R\) are respectively the lateral and torsional stiffness matrices of order \(N\) about the elastic center \(K_L, K_R\) = the diagonal mass matrix, \(\ddot{\phi}_L\) and \(\ddot{\phi}_R\) = lateral and torsional components of the mode shapes, \(\omega\) = the natural frequency, \(e = \) eccentricity of mass center \(M_C\) from \(K_L\), and \(i = \) mass radius of gyration about \(K_L\), assumed to be constant. Methods of assembling \(K_L\) and \(K_R\) from the individual structural assemblages of beams, columns and walls are well known (e.g.,4,19) and will not be discussed here. As will be shown subsequently, under certain conditions the \(2N \times 2N\) eigenproblem in Eq. 1 may be uncoupled into \(N \times 2\)-DOF systems, thereby considerably simplifying the problem. Let \(\omega_{Lo}\) and \(\ddot{\phi}_{Lo}\) be the solutions of the eigenproblem:

\[(K_L - \omega^2 M) \ddot{\phi}_{Lo} = 0\]  
(2)

and let \(\omega_{Ro}\) and \(\ddot{\phi}_{Ro}\) be the solution of

\[(K_R - \omega^2 M) \ddot{\phi}_{Ro} = 0\]  
(3)

in which \(\omega_{Lo}, \omega_{Ro}, \ddot{\phi}_{Lo}\) and \(\ddot{\phi}_{Ro}\) are the natural frequencies and mass normalized node shapes respectively of the uncoupled \(N \times N\) systems of Eqs. 2 and 3. Expressing the normal node shapes \(\ddot{\phi}\) (Eq. 1) in terms of the modal coordinates \(\ddot{\phi}_{Lo}\) and \(\ddot{\phi}_{Ro}\):
\[
\begin{bmatrix}
\vec{ \gamma }_L \\
\vec{ \gamma }_B
\end{bmatrix} =
\begin{bmatrix}
\Phi_{L0} & 0 \\
0 & \Phi_{B0}
\end{bmatrix}
\begin{bmatrix}
\vec{ \gamma }_L \\
\vec{ \gamma }_B
\end{bmatrix} = \Phi_c \vec{ \gamma }_D
\]

(4)

in which \( \Phi_{L0} \) and \( \Phi_{B0} \) are the modal matrices of \( \Phi_{L0} \) and \( \Phi_{B0} \), substituting in Eq. 1 and premultiplying by \( \Phi_c^T \) (\( T \) designates the transpose) using the orthogonality properties, the following expression is obtained:

\[
\begin{bmatrix}
\omega_{L0}^2 & 0 \\
0 & \omega_{B0}^2
\end{bmatrix} - \omega^2 = \begin{bmatrix}
I & \Phi_{L0}^T \cdot N \cdot \Phi_{B0} \cdot s^{1/2} \\
\Phi_{B0}^T \cdot M \cdot \Phi_{L0} \cdot s^{1/2} & I \left[ \frac{1}{4} + \frac{(s^{1/2})^2}{4} \right]
\end{bmatrix}
\begin{bmatrix}
\vec{ \gamma }_D \\
\vec{ \gamma }_R
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(5)

in which \( \omega_{L0}^2 \) and \( \omega_{B0}^2 \) are the diagonal matrices of \( \omega_{L0}^2 \) and \( \omega_{B0}^2 \), respectively. To uncouple Eq. 5 into \( N \) 2-DOF equations it is required that:

\[
\Phi_{L0} = \Phi_{B0} \cdot 1
\]

(6)

so that the off-diagonal matrices in the second matrix of Eq. 5 become themselves diagonal. This condition is satisfied when the matrices \( K_B \) and \( K_R \) commute, or, mathematically:

\[
K_L K_R = K_R K_L
\]

(7)

Then Eq. 5 takes the form:

\[
\begin{bmatrix}
\omega_{L0}^2 & 0 \\
0 & \omega_{B0}^2
\end{bmatrix} - \omega^2 = \begin{bmatrix}
I & I \cdot s^{1/2} \\
I \cdot s^{1/2} & I \left[ 1 + \frac{(s^{1/2})^2}{4} \right]
\end{bmatrix}
\begin{bmatrix}
\vec{ \gamma }_L \\
\vec{ \gamma }_R
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(8)

and it is readily seen that Eq. 8 is equivalent to \( N \) 2 \times 2 matrix equations of the form:

\[
\begin{bmatrix}
\omega_{Lj}^2 & 0 \\
0 & \omega_{Bj}^2
\end{bmatrix} - \omega_j^2 = \begin{bmatrix}
1 & s^{1/2} \\
s^{1/2} & 1 \left[ 1 + \frac{(s^{1/2})^2}{4} \right]
\end{bmatrix}
\begin{bmatrix}
\vec{ \gamma }_{Lj} \\
\vec{ \gamma }_{Rj}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(9)

which upon solution of the quadratic characteristic equation yields the natural frequencies:
\[
\frac{\omega_n^2}{\omega_{Lo}^2} = \frac{1}{\mathbf{I}} \left[ 1 + \left( \frac{\omega_n^2}{\omega_{Lo}^2} \right)^2 \right] \mathbf{I} + \frac{3}{\mathbf{I}} \left[ \frac{\omega_n^2}{\omega_{Lo}^2} \right]^2 \mathbf{I} + \mathbf{I} \left( \frac{\omega_n^2}{\omega_{Lo}^2} \right)^3 \mathbf{I}
\]

(10)

and the mode shapes

\[
\begin{align*}
\psi_{n}^{L} & = \frac{1}{\left[ 1 - \left( \frac{\omega_n^2}{\omega_{Lo}^2} \right)^2 \right]^{\frac{1}{2}} (\frac{\omega_n^2}{\omega_{Lo}^2})^\frac{1}{2} (\frac{\omega_n^2}{\omega_{Lo}^2})^\frac{3}{2}} \\
\psi_{n}^{R} & = \frac{1}{\left[ 1 - \left( \frac{\omega_n^2}{\omega_{Lo}^2} \right)^2 \right]^{\frac{1}{2}} (\frac{\omega_n^2}{\omega_{Lo}^2})^\frac{1}{2} (\frac{\omega_n^2}{\omega_{Lo}^2})^\frac{3}{2}} 
\end{align*}
\]

(\(n = 1, 2\))

(11)

Eq. 9 is the 2-DOF model used by Skinner et al (17), Rosebluth and Blanchard (15) and Fauszi (13, 14), and it is evident that it is strictly applicable to eccentric multistory buildings in which the mass centers and the elastic centers are located along two vertical axes only when Eq. 7 is satisfied.

Although in general Eq. 6 is not exactly satisfied, so that \(\mathbf{M} \cdot \mathbf{\ddot{x}} = \mathbf{K} \cdot \mathbf{x}\) and

\[
\mathbf{K} \cdot \mathbf{x} = \mathbf{M} \cdot \mathbf{\ddot{x}}
\]

are not diagonal, yet an approximate solution may be obtained by neglecting the off-diagonal terms of these matrices. Before discussing the accuracy of this approximation it will be shown that the behavior of the second model described earlier in this section may also be expressed in the form of Eq. 5 and thus be amenable to the same approximate solution.

Using EC as the reference axis the free vibration equation for the second model reads:

\[
\begin{pmatrix}
\mathbf{K} & \mathbf{S} \\
\mathbf{S}^T & \mathbf{M} \cdot \mathbf{e} + \mathbf{K} \cdot \mathbf{e} + \mathbf{S} \cdot \mathbf{e}
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_L \\
\mathbf{v}_R
\end{pmatrix}
= \begin{pmatrix}
\mathbf{0} \\
\mathbf{0}
\end{pmatrix}
\]

(12)

in which \(\mathbf{S}\) = lateral stiffness matrix of the second framing system, \(a = \) distance from its elastic center to KC, \(r_K\) and \(r_E\) = radii of gyration of \(\mathbf{K}_L\) and \(\mathbf{S}_L\) about EC and SC respectively (Fig. 2). Now it is possible to transform Eq. 12 into the form of Eq. 1 by means of the eigenvalues \(\omega_n^2\) and the eigenvectors of the 2-DOF system (6):
\[
\begin{bmatrix}
1 & \frac{a}{2k} \\
\frac{a}{2k} & \left(\frac{r_0}{r_k}\right)^2 + \left(\frac{r_0}{r_k}\right)^2
\end{bmatrix}
- \frac{1}{2}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
\gamma
\end{bmatrix}
= \begin{bmatrix}
0 \\
\varepsilon
\end{bmatrix}
\]
(13)

which is:
\[
\begin{bmatrix}
\bar{F}_L \\
\bar{F}_K
\end{bmatrix}
= \frac{1}{1 + \gamma^2}
\begin{bmatrix}
1 & -1 \cdot \gamma \\
1 \cdot \gamma & 1
\end{bmatrix}
\begin{bmatrix}
\phi_L \\
\phi_K
\end{bmatrix}
= \Gamma \begin{bmatrix}
\phi_L \\
\phi_K
\end{bmatrix}
\]
(14)

which in turn leads to:
\[
(\bar{F}^2 - \omega^2 \bar{K}) \phi = 0
\]
(15)

or:
\[
\begin{bmatrix}
\bar{K}_L + \frac{r_0^2}{2k} & 0 \\
0 & \bar{K}_K + \frac{r_0^2}{2k}
\end{bmatrix}
- \omega^2
\begin{bmatrix}
N^* & \psi^* \\
\psi^* & \psi^* \left(\gamma^2 + \frac{2a^2}{r_k^2}\right)
\end{bmatrix}
\begin{bmatrix}
\bar{F}_L \\
\bar{F}_K
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
(15a)

in which
\[
\begin{align*}
p_{L2}^2 &= \frac{1}{4} \left[ 1 + \left(\frac{a}{r_k}\right)^2 + \left(\frac{r_0}{r_k}\right)^2 \right] \left[ \left(\frac{r_0}{r_k}\right)^2 - \left(\frac{r_0}{r_k}\right)^2 \right] + 4 \left(\frac{r_0}{r_k}\right)^2 \left(\frac{a}{r_k}\right)^2 \\
\gamma &= -(1 - p_{L2}/(a/r_k)) \left(\frac{r_0}{r_k}\right)^2
\end{align*}
\]
(16)

\[
N^* = \pi \left\{ \left(\gamma^2 + \frac{2a^2}{r_k^2}\right) + \left[1 - \gamma + \frac{a^2}{r_k^2}\right] \right\} / (1 + \gamma^2)
\]
(16)

\[
\psi^* = \gamma \left[ \frac{a^2}{r_k^2} + 2 \right] / (1 + \gamma^2)
\]
(16)

\[
e^* = \left[ \frac{a^2}{r_k^2} - (1 - \gamma) \right] / (1 + \gamma^2)
\]
(16)

It is evident that Eq. 15 has the same form as Eq. 1, hence it may be transformed into
Eq. 5. In this case it is unlikely that:
\[
(K_L + \frac{r_0^2}{2k})(K_L + \frac{r_0^2}{2k}) = (K_L + \frac{r_0^2}{2k})(K_L + \frac{r_0^2}{2k})
\]
(16a)
since by definition the two structural systems have different stiffness properties, so that it might appear that the two matrices \( \begin{bmatrix} M_x^T & 0 \\ 0 & M_y^T \end{bmatrix} \) and \( \begin{bmatrix} M_x^T M_x & 0 \\ 0 & M_y^T M_y \end{bmatrix} \) may not be predominantly diagonal. However, a large number of building structures analyzed by the authors showed that neglecting the off-diagonal terms in both models did not appreciably affect the natural frequencies and mode shapes. This is to be expected, since the corresponding node shapes of the two uncoupled systems, although different, are sufficiently close to render the diagonal terms larger than the off-diagonal ones. It therefore appears that the 2-DOF model in Eq. 9 is sufficiently accurate for most practical purposes. Yet as with every approximate procedure, estimates of the expected errors are required. Such estimates may be obtained by means of a perturbation procedure (1) which is outlined in Appendix I.

**CALCULATION OF THE DYNAMIC EARTHQUAKE RESPONSE**

Under earthquake excitation, the equation of motion associated with Eq. 15 or Eq. 1 may be written as

\[
\ddot{\mathbf{\eta}} + \mathbf{C} \dot{\mathbf{\eta}} + \mathbf{K} \mathbf{\eta} = -\mathbf{M} \ddot{\mathbf{\eta}}_{\text{ext}} + \mathbf{F} \mathbf{\eta}_0
\]

(17)

in which \( \mathbf{\eta} \) is the displacement response vector of order 2 N, \( \mathbf{C} \) = the damping matrix assumed to be proportional (3), \( \mathbf{\eta}_0 \) = the unidirectional earthquake acceleration time history, \( \mathbf{\eta}\_g \) is a vector of ones and zeros with the ones corresponding to the degree of freedom in the direction of \( \mathbf{\eta}_g \), \( \Gamma_0 \) = the static modal participation matrix as defined in Eq. 14.

Note that for the first model in which the system is statically uncoupled, \( \Gamma_0 \) becomes a unit diagonal matrix.

The response \( \mathbf{\eta} \) could be computed by a step-by-step integration of the matrix Eq. 17, and several techniques are available for this purpose (e.g. ref. 5). However, the normal mode approach combined with the response spectrum technique is much simpler and is sufficiently accurate for many practical applications. Following the normal mode procedure outlined in the preceding section, the displacements are first expressed
terms of the uncoupled modal coordinates:

\[ \ddot{\vec{q}} = \ddot{\bar{\Omega}}_0 \cdot \vec{q} \]  

(18)

in which \( \vec{q} \) is the vector of nodal coordinates amplitudes. Then the approximate expression for the \( N \times 2 \times 2 \) equations of motions are obtained as follows:

\[
\begin{bmatrix}
1 & e^{s^i/\lambda^i} \\
\lambda^i & \lambda^i e^{s^i/\lambda^i}^2
\end{bmatrix}
\begin{bmatrix}
\xi_{LJ} \\
\xi_{RJ}
\end{bmatrix}
+
\begin{bmatrix}
2 \Omega_{LJ} \xi_{LJ} & 0 \\
0 & 2 \Omega_{RJ} \xi_{RJ}
\end{bmatrix}
\begin{bmatrix}
\xi_{LJ} \\
\xi_{RJ}
\end{bmatrix}
+
\begin{bmatrix}
\Omega_{LJ}^2 & 0 \\
0 & \Omega_{RJ}^2
\end{bmatrix}
\begin{bmatrix}
\xi_{LJ} \\
\xi_{RJ}
\end{bmatrix}
= \begin{bmatrix}
\Gamma_{LJ} \\
\Gamma_{RJ}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_N \\
\ddot{u}_R
\end{bmatrix}
\]

(19)

in which:

\[
\begin{bmatrix}
\Gamma_{LJ} \\
\Gamma_{RJ}
\end{bmatrix}
= \begin{bmatrix}
\bar{\phi}_{LJ}^T \Omega_{LJ} \bar{\phi}_{LJ}^-T \\
\bar{\phi}_{RJ}^T \Omega_{RJ} \bar{\phi}_{RJ}^-T
\end{bmatrix}
\]

(20)

is the vector of the nodal participation factors. Note that, for \( \bar{\phi}_{LJ} \equiv \bar{\phi}_{RJ} \equiv 1 \), Eq. 19 is exact and moreover, for the first model, where \( \Gamma_{n} \equiv 1 \), the right hand side becomes:

\[
\begin{bmatrix}
\Gamma_{LJ} \\
\Gamma_{RJ}
\end{bmatrix}
= \begin{bmatrix}
\Gamma_{LJ} \\
\Gamma_{RJ} e^{s^i/\lambda^i}
\end{bmatrix}
= \begin{bmatrix}
1 \\
e^{s^i/\lambda^i}
\end{bmatrix}
\]

(21)

in which \( \Gamma_{LJ} \) is the lateral modal participation factor. The 2-DOF equation of motion given in Eq. 19 with its right hand side as given by Eq. 21 can be solved numerically for the prescribed acceleration \( \ddot{u}_N \) for any set of \( \Omega_{LJ}, \Omega_{RJ}, L_{LJ}, L_{RJ} \) and \( e^{s^i/\lambda^i} \) to yield the maximum 2-DOF response. Their maxima can then be combined in the RSS (root-sum-square) manner to obtain an estimate of maximum response of the system. This constitutes the 2-DOF response spectra procedure proposed by Fenzien (13,14). Since such spectra are not readily available, the standard 1-DOF response spectra are used and the estimate of the maximum 2-DOF response is obtained by modified RSS formula proposed by Rosenbluth and Glordy (15). For this purpose Eq. 19 is uncoupled by means of the nodal
matrix of Eq. 11, which leads to:

\[ \ddot{q}_{n,j} + 2\omega_{n,j} \cdot \dot{q}_{n,j} + \omega_{n,j}^2 q_{n,j} = \Gamma_{n,j} \ddot{u}_g \quad (n = 1, 2) \]  

where \( \Gamma_{n,j} \) are given by Eq. 20 and:

\[ \begin{bmatrix} \Gamma_{11} \\ \Gamma_{12} \\ \Gamma_{21} \\ \Gamma_{22} \end{bmatrix} = \Psi^T \begin{bmatrix} \Gamma_{11} \\ \Gamma_{21} \\ \Gamma_{12} \\ \Gamma_{22} \end{bmatrix} \]  

where \( \Psi^T \) is the approximate 2-DOF modal matrix given in Eq. 11. The solution of each independent modal response equation (Eq. 22) may be written as:

\[ q_{n,j} = \Gamma_{n,j} S_{dn,j} \quad (n = 1, 2; j = 1, N) \]  

where \( S_{dn,j} \) is the spectral displacement of \( \ddot{u}_g \).

To obtain the modal lateral forces \( P \) and the modal torsional moments \( T \) at any reference level \( k \) of the structure, backward transformation is required, which leads to:

\[ \begin{bmatrix} P_k \\ T_k \end{bmatrix} = \mathbf{M}_s^{-1} \Phi_0^T \begin{bmatrix} \Gamma_{11} \\ \Gamma_{21} \\ \Gamma_{12} \\ \Gamma_{22} \end{bmatrix} S_{dn,j} \]  

where \( S_{ak} \) is the spectral acceleration. Explicitly, the components of the coupled \( j \)-th pair of the lateral forces are:

\[ \begin{bmatrix} P_{1j} \\ P_{2j} \end{bmatrix} = \begin{bmatrix} M_s^{K^1} \phi^K_{Lo,j} [\phi^1_{L,j1} \cdot \phi^K_{j} (\cdot)^*] \Gamma_{11} S_{aj} \\ M_s^{K^2} \phi^K_{Lo,j} [\phi^1_{L,j2} \cdot \phi^K_{j} (\cdot)^*] \Gamma_{22} S_{aj} \end{bmatrix} \]  

in which \( \phi^K_{j} \) is proportional to the ratio of \( k \)-th components of the noncoupled modal shapes and reads:

\[ \phi^K_{j} = \frac{1}{\phi^K_{Lo,j}} \phi^K_{j} \]  

and the torsional moments are given by:
\[
\begin{align*}
\begin{bmatrix}
\gamma_1^k \\
\gamma_2^k
\end{bmatrix}
&= 
\begin{bmatrix}
\phi_{K1}^L (\psi_1 (s'/s) + \phi_{K1}^L (s'/s + 1))
& \phi_{K1}^L \\
\phi_{K2}^L (s'/s') + \phi_{K2}^L (s'^2/s^2 + 1) & \phi_{K2}^L
\end{bmatrix}
\begin{bmatrix}
\Delta_{11} \\
\Delta_{12}
\end{bmatrix}
\end{align*}
\]

(28)

For the static method of seismic analysis, several building codes (7) require that the dynamic eccentricity due to nodal coupling be computed. Denoting the dynamic eccentricity by \( e_d^* \) and the static eccentricity by \( e^* \), the corrected 2-DOF RSS (15) amplification of the static eccentricity is given by:

\[
C_j = \frac{e^*_d}{e^*} = \frac{1}{e^*} \left[ \frac{\gamma_1^2 + \gamma_2^2 - 2 \gamma_1 \gamma_2 (1 + \frac{\gamma_2^2}{\gamma_2^2})}{\gamma_1^2 + \gamma_2^2 + 2 \gamma_1 \gamma_2 (1 + \frac{\gamma_2^2}{\gamma_2^2})} \right]^{\frac{1}{2}}
\]

(29)

in which the correction coefficient for close frequencies \( \xi_{12j} \), proposed by Rosenbluth and Blodgett (15), is:

\[
\xi_{12j} = \frac{\omega_{1j}^2 - \omega_{2j}^2}{\omega_{1j}^2 + \omega_{2j}^2}
\]

(30)

where \( \omega \) designates the damped natural frequencies.

The dynamic amplification factor \( C_j \) was evaluated numerically as a function of \( \lambda_L / \psi_{Ro} \) for a large number of building structures. Since in general \( \psi_{L1}^K \) is not constant but varies along the building height, its values were computed at a level where \( \psi_{Ro}^K = 1 \), i.e., where \( \psi_{Ro}^K = \psi_{Ro}^K \). Fig. 3 shows the dependence of \( C_j \) on the coupling parameter \( \psi''/\psi' \), for the first mode, using the spectrum shown in the insert. The continuous heavy line represents the exact 2-DOF case and the dashed band shows the range of the results for the structures analyzed. These structures were chosen so as to produce large variations between their uncoupled lateral and torsional mode shapes. This was achieved by choosing uniform flexural cantilever properties for the one, and uniform shear cantilever properties for the other. It should be noted that the boundaries of the bands were plotted based on the exact and on the approximate solutions of the problems analyzed, so that the narrowness of the
hands should give confidence in the proposed approximate solution. Fig. 4 reveals an interesting feature of eccentric systems, namely the fact that while torsional coupling leads to the magnification of static eccentricity, it tends to lower the lateral shear forces. This phenomenon was first reported by Rosenblueth and Klordy (15) for single story structures. No graphs are presented for the amplification factor of the second model. However, Figs. 3 and 4 may still be used with a suitable correction which is given in Appendix II.

As already noted, another interesting feature of a structure in which \( \frac{\sum F_0}{\sum F_0} \) \( \frac{1}{i} \) is the variation of the dynamic amplification factor \( C_j^k \) along the building height. As can be seen from Eqs. 26 and 29, \( C_j^k \) does not depend on \( \alpha_j^k \) when \( e/i \) is large, and remains constant along the building height. For small values of \( e/i \), the variation of \( C_j^k \) with height is proportional to the mode shape ratio \( \alpha_j^k \). This is of some practical importance, since an approximation for \( C_j^k \) at any level may be obtained by interpolating between the value of \( C_j \) as given in Fig. 3 and \( C_j \cdot \alpha_j^k \). This can be seen in Fig. 6.

In the following section two numerical examples are presented to illustrate the application of the proposed method.

**Numerical Examples**

**Example I** - The 43-story Wells Fargo building in San Francisco originally analyzed by Nedearis (11) is used to illustrate the procedure (Fig. 5). The relevant structural properties are given below:

\[
K_x = K_y = 110 \text{ kip/ft} \quad (1.610 \times 10^6 \text{ N/m}) ; \quad K_R = 495 \text{ 144000 kip-ft/radian} \quad (671.83 \times 10^3 \text{ N-m/rad}) ;
\]

\[
M = 73.56 \text{ kip sec}^2/\text{ft} \quad (1.073 \times 10^6 \text{ kg}) ;
\]

\[
i = 62.99 \text{ ft} \quad (19.20 \text{ m}) ;
\]

\[
e_x = e_y = 2.03 \text{ ft} \quad (0.619 \text{ m}) ;
\]

\[
K_x = K_y = K_y ;
\]
The parameters required for the analysis are easily obtained from the above properties:

e/ι = 0.0456 ;  \; \; \; \; \; Ω_{Lo}^{j} = ι_{K}/ι = 1.06

Free Vibrations. - Since the structural properties of this building satisfy Eq. 1 and Eq. 7 the 129-DOF system can be reduced, due to symmetry, to a 45-DOF system plus one 2-DOF system. The 43-DOF problem may then be further reduced to a single parameter continuous system if it is recognized that the structure is practically an uniform shear beam whose dynamic properties are well known (12). To obtain the natural frequencies Eq. 10 is used:

\[
\begin{aligned}
\left( \frac{ω^{2}}{ω_{Lo}^{j}} \right) _{j} &= \frac{1}{4} (1 + 1.06^{2} + 0.0456)^{2} \pm \sqrt{(1 - 1.06^{2} + 0.0456)^{2} + 4 \cdot (0.0456 - 1.06)^{2}} \\
&= 0.9959 \\
&= 1.1476
\end{aligned}
\]

so that for $\omega_{Lo}^{1} = 1.398$ rad/sec and $\omega_{Lo}^{2} = 4.192$ rad/sec:

$\omega_{1} = 1.398 \sqrt{0.9959} = 1.398$ rad/sec (1.398)

$\omega_{2} = 1.398 \sqrt{1.1476} = 1.498$ rad/sec (1.498)

$\omega_{3} = 4.192 \sqrt{0.9959} = 4.162$ rad/sec (4.165)

$\omega_{4} = 4.192 \sqrt{1.1476} = 4.491$ rad/sec (4.491)

The values in parentheses are those computed by Medearis from his 129-DOF system. Using Eq. 11 and Eq. 4 the coupled mode shapes are evaluated to yield:

\[
\phi_{j}^{i} = \left\{ \begin{array}{c}
0.9541 \\
0.2926
\end{array} \right\} \quad \phi_{2}^{j} = \left\{ \begin{array}{c}
0.1566 \\
-0.9425
\end{array} \right\} \quad j = 1 \cdots 43
\]

Higher natural frequencies and their coupled mode shapes may be evaluated in a similar manner.
Forced Response. – The modal forces and torques are evaluated using Eqs. 21, 26 and 28:

\[
\begin{align*}
\{P_1\} &= \begin{bmatrix} 0.9365 S_{a1} \\ 0.1294 S_{a2} \end{bmatrix} \mathcal{F}_{L0J} \mathcal{F}_{LxJ} \\
\{P_2\} &= \begin{bmatrix} -0.3327 S_{a1} \\ -0.1865 S_{a2} \end{bmatrix}
\end{align*}
\]  
(Eq. 26)

\[
\begin{align*}
\{T_1\} &= \begin{bmatrix} 0.3327 S_{a1} \end{bmatrix} \mathcal{F}_{L0J} \mathcal{F}_{LxJ} \\
\{T_2\} &= \begin{bmatrix} -0.1865 S_{a2} \end{bmatrix}
\end{align*}
\]  
(Eq. 28)

Substituting these values in Eq. 29, the amplification of the static eccentricity for the spectrum given in Fig. 3 is obtained:

\[
C_J = \left( \frac{S}{e} \right)_J = \frac{1}{0.0456} \left[ \frac{0.3327^2 \cdot 0.1065^2 - (S_{a1}/S_{a2})^2 }{0.9365^2 \cdot 0.0129^2 + 0.9365 \cdot 0.0129 \cdot (S_{a1}/S_{a2})} \right]^{1/2} \]  
(Eq. 29)

which for the first pair of coupled modes is:

\[
C_J = \left( \frac{S}{e} \right)_1 = \frac{1}{0.0456} \left[ \frac{0.0996}{0.0850} \right]^{1/2} = 7.35
\]

For the higher pairs, the ratios of the spectral values are different, but the amplification factors remain practically unchanged.

The ratio of the computed lateral force \( F \) to the uncoupled lateral force \( F_0 \) for the first pair is given by:

\[
\left( \frac{F}{F_0} \right)_1 = \sqrt{0.0850 \cdot \frac{S_{a1}}{S_{a2}}} = 0.5367
\]

This ratio is practically unchanged for the higher modes, and it is evident that the effect of coupling on these forces is quite small. It follows that the RSS eccentricity magnification factor \( C \) as well as the RSS lateral force ratio \( F/F_0 \) is quite accurately given by the results for the first pair of modes as given above. The dynamic eccentricity
is found to be:

\[ e_d = 7.35 \cdot 0.9387 \cdot 2.87 = 19.79 \text{ ft (6.03 m)} \]

Comparing this result with a typical code provision as given by:

\[ e_d = 1.5a + 0.05 L_{\text{max}} = 12.15 \text{ ft (3.70 m)} \]

It is seen that in this case even when the accidental eccentricity effect (the 0.05 \( L_{\text{max}} \) term) is taken into consideration, code provisions substantially underestimate the amplifica-
tion of eccentricity.

**Example II** - A twenty story wall-frame building whose floor plan is given in Fig. 2 is used to illustrate the analysis procedure for the second model. The story height is 3.00 m (9.84 ft) and the total height \( H = 60.00 \) m (196.85 ft). The stiffness properties of the frames and the walls are given in Table I, \( E = 2.5 \times 10^7 \text{ KN/m}^2 (5.6 \times 10^6 \text{ psi}) \), and the story mass \( M = 274,000 \text{ kg (18.78 kip sec}^2/\text{ft}) \).

The parameters required for the calculations are:

\[ a/i = e/( (L_x^2 + L_y^2)/12)^{1/2} = 4.94/10.08 = 0.49 \]

\[ r_{x,i} = \frac{\sum (I_{x,j} r_{x,j}^2 + I_{y,j} r_{y,j}^2 + L_{x,j})/ \sum I_{x,j}}{\sum r_{x,j}^{3/2}} = 12.86/10.08 = 1.28 \]

\[ a/r_{K} = 8.94/12.86 = 0.70 \]

\[ r_{y,i} = \frac{\sum (I_{y,j} r_{y,j})^{3/2}}{\sum I_{y,j}} \]

\[ r_{y,K} = \frac{\sum (I_{y,j} r_{y,j}^2)}{\sum I_{y,j}} \]

Using the relations in Eq. 15, the transformed parameters are found to be:

\[ P_{1,2} = 1.245 ; \quad 0.356 \quad \gamma = -0.799 ; \quad K = 350,347,36 \text{ kg (24.012 kip sec}^2/\text{ft}) \]

\[ i^0 = 1.637 ; \quad e^0 = 0.0437 \]

The natural frequencies of the two noncoupled systems are given in Table II.
The dynamic properties and the response of the structure to earthquake excitation (using the spectrum in Fig. 3) were computed by the proposed approximate method and by a standard eigenvalue computer program.

The dynamic properties of the structure for the first ten modes are given in Table III, the results under "app." are those obtained by the proposed method, whereas the values under "exact" are those calculated using the lumped mass computer program. Comparing the results for the natural frequencies, modal participation factors and mode shapes, it is evident that the agreement is quite satisfactory. The big error in the shape of mode 9 is of no consequence since it is due to a big error occurring at a zero-crossing of the mode shape.

The amplification factors and the lateral force ratios are given in Table IV for the first five pairs of coupled modes. Again the agreement between the proposed method and the exact solution is quite satisfactory.

The variation of the eccentricity amplification factor along the height is of some interest. As mentioned in the foregoing section, this variation of the exact value is bracketed by $C_j$ and by $C_j \cdot (1)^k$. This is illustrated in Fig. 6 for the first pair of coupled modes ($w_1$).

SUMMARY AND CONCLUSIONS

An approximate method has been presented for the evaluation of the dynamic properties and the seismic response of a class of topologically coupled multi-story buildings. The solution is based on the similarity between the uncoupled lateral and torsional mode shapes which leads to a 2-degree-of-freedom representation of the coupling effect. The uncoupling of the system into two N-DOF systems plus N 2-DOF systems is likely to reduce the complexity of computation substantially. An extensive parametric study has shown that the error
DYNAMIC TORSIONAL COUPLING IN ASYMMETRIC BUILDING STRUCTURES

KEY WORDS: Buildings; Dynamics; Earthquakes; Structural Engineering; Tall Buildings; Torsional Coupling; Vibrations.

ABSTRACT: An approximate method is proposed for the dynamic analysis of torsionally coupled tall building structures by utilizing the properties of their uncoupled counterparts. An exact solution satisfying the uncoupling requirements is first given for the particular case in which the lateral and torsional stiffness matrices are uncoupled by some transformation. The method is then applied to a wider class of structures where this condition is only approximately satisfied reducing the dynamic coupling problem to an approximate two-degree-of-freedom system. Simple formulae and graphical representations of dynamic magnification of static eccentricity are given permitting a rapid evaluation of seismic response. Two numerical examples illustrate the use of the proposed method, checking on its accuracy and comparing its results with seismic code provisions.
DYNAMIC TORSIONAL COUPLING IN ASYMMETRIC BUILDING STRUCTURES

By Andrei Reinhold, (1) Avigdor Butenberg M. ASCB (2) and Jacob Gluck (3)

INTRODUCTION

The torsional response of building structures to earthquake excitation is of increasing concern to structural engineers. Three main torsional effects are well recognized: accidental eccentricity, torsional ground motion and coupling between lateral and rotational vibrations. This last effect which characterizes buildings with asymmetric layout of the structural components with respect to the floor plan is the subject of this study.

The behavior of single story asymmetric structures under lateral earthquake excitation has been extensively studied. Apparently, Boussen and Gutin (6) were the first to point out that the static method of analysis underestimates significantly the maximum stresses in such structures. The effect of coupling, especially when the lateral and torsional frequencies are close, was investigated by Bustamente and Rosenblueth (2), Skinner et al (18), Rosenblueth and Blordoy (15), Keintzel (10) and very recently by Kan and Chopra (9). These studies led to the realization that if the static methods of analysis are to continue in use, statically computed eccentricities should be amplified. Accordingly several seismic codes (7) adopted simple formulas to allow for this effect. The limitations inherent in the RSS (root-sum-square) formula for combining the spectral values in the modal analysis (e.g., 12) were

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involved using the method is sufficiently small and may be considered acceptable for earthquake analysis. Moreover, a check on the accuracy of the solution may be easily made by means of a perturbation analysis as outlined in Appendix I.

The examples worked out demonstrate the limitation of the static methods of analysis for estimating the amplification of dynamic eccentricity. If such methods are to continue in use, the magnification factors stipulated by building codes should be made dependent on some basic coupling parameters. It has been shown that the two degree-of-freedom approximation as presented herein is likely to provide such parameters for a relatively large class of building structures.

**APPENDIX I. - PERTURBATION ANALYSIS**

Eq. 5 may be rewritten in the form:

\[
\begin{bmatrix}
\omega_0^2 - \omega_\lambda \\
-\frac{\omega_0^2}{\omega_\lambda} \\
\frac{\omega_0^2}{\omega_\lambda}
\end{bmatrix}
\begin{bmatrix}
I & I & \frac{a}{a} \\
I & I & \frac{a}{a} \\
\frac{a}{a} & I & \frac{a}{a} \\
\end{bmatrix}
- \epsilon \lambda
\begin{bmatrix}
\frac{\alpha}{\omega_0} - \frac{\beta}{\omega_0} - (1) \frac{e}{a} \\
\frac{\alpha}{\omega_0} - \frac{\beta}{\omega_0} - (1) \frac{e}{a} \\
\frac{\alpha}{\omega_0} - \frac{\beta}{\omega_0} - (1) \frac{e}{a}
\end{bmatrix}
\begin{bmatrix}
\vec{\psi}_L \\
\vec{\psi}_R
\end{bmatrix}
= \begin{bmatrix}
\vec{0} \\
\vec{0}
\end{bmatrix}
\]

or

\[
(\omega_0^2 - \lambda h_\lambda - \epsilon \lambda \Delta \beta) = 0
\]

in which \( \lambda = \omega_0^2 \) is the eigenvalue and \( \epsilon \) is a perturbation factor introduced in the above expression to facilitate grouping of terms with comparable degrees of approximation \((1)\). Comparing Eq. 31 with Eq. 8 shows that the unperturbed solutions of Eq. 31 are those of Eq. 6\(^{(*)}\). Consider a correction to a particular eigenvalue \( \lambda_j = \omega_j^2 \) and eigenvector \( \vec{\psi}_j \) in terms of the powers of \( \epsilon \):

\[\text{(*) or similar in form if the diagonal term rather than } I \frac{e}{a} \text{ are retained in the unperturbed solution - such an approximation leads to somewhat better results.}\]
\[ \lambda_j = \lambda_j^{(0)} + \mathcal{E} \lambda_j^{(1)} + \mathcal{E}^2 \lambda_j^{(2)} \]  

(32)

\[ \vec{f}_j = \vec{f}_j^{(0)} + \mathcal{E} \vec{f}_j^{(1)} + \mathcal{E}^2 \vec{f}_j^{(2)} \]  

(33)

Substituting Eq. 32 and Eq. 33 into Eq. 31 and equating like powers of \( \mathcal{E} \), a system of perturbation equations is obtained (e.g. (1)) of which the first order one is given by:

\[ (\vec{f}_j^{(0)})^T \lambda_j^{(0)} \vec{f}_j^{(1)} = \lambda_j^{(1)} \vec{f}_j^{(0)} \]  

(34)

The first order correction of the eigenvector is now expressed as a linear combination of all the unperturbed eigenvectors and reads:

\[ \vec{f}_j^{(1)} = \sum_{i=1}^{N} a_{ij}^{(1)} \vec{f}_j^{(0)} \]  

(35)

Note that the two sums represent the two vectors given in Eq. 11 and that each has only two non-zero elements. Substituting Eq. 35 in Eq. 34 premultiplying by \( \vec{f}_j^{(0)T} \) and using the orthogonality properties of the normal modes, one obtains \( \lambda_j^{(1)} \) and \( a_{ij} \) as follows:

\[ \lambda_j^{(1)} = -\lambda_j^{(0)} \vec{f}_j^{(0)T} \Delta \mu \vec{f}_j^{(0)} \]  

(36)

and

\[ a_{ij} = \frac{\lambda_j^{(0)}}{\lambda_i^{(0)} - \lambda_j^{(0)}} \vec{f}_i^{(0)T} \Delta \mu \vec{f}_j^{(0)} \]  

(37)

Note that \( a_{ij} \) is still undetermined. It is determined in the final solution by normalizing \( \vec{f}_j^{(1)} \).

To obtain a second correction it is necessary to perform a second perturbation analysis. This again requires the expression of the eigenvector \( \vec{f}_j^{(2)} \) in terms of the unperturbed vector \( \vec{f}_j^{(0)} \). For details the reader may consult reference (17) in which an analogous procedure for a continuous structure is given.
APPENDIX II. - AMPLIFICATION OF STATIC ECCENTRICITY FOR SECOND MODEL

Comparing the right hand side of Eq. 21 with that of Eq. 20 it is seen that new loading terms are added to the second model. The modal participation factor \( \Gamma \), then, takes the form:

\[
\Gamma = \Gamma_a - \gamma 1^* \Gamma_b
\]

in which \( \Gamma_a \) and \( \Gamma_b \) are respectively dependent on the transformed 'internal' and 'rotational' components of the excitation.

Using Eqs. 26, 28 and 29 the eccentricity amplification factor \( c_j \) for the second model may be expressed as a function of \( c_j \) (given in Fig. 3) to read:

\[
c_j = \frac{1 + (\gamma \frac{1^2}{c_j^*} e^*)^2 \cdot \sum_{i=1}^{2} \gamma_i^4 \left( \frac{\gamma_i^2 - 2 \gamma_i^2}{\gamma_i^2 + \gamma_i^2} \right) \cdot \frac{\alpha_i^2}{\gamma_i^2} \cdot \frac{\gamma_i^2}{\alpha_i^2} \cdot \sum_{i=1}^{2} \frac{\gamma_i^4}{\gamma_i^2} \cdot \frac{\gamma_i^2}{\gamma_i^2}}{1 + (\gamma \frac{1^2}{c_j^*} e^*)^2 \cdot \sum_{i=1}^{2} \gamma_i^4 \left( \frac{\gamma_i^2 - 2 \gamma_i^2}{\gamma_i^2 + \gamma_i^2} \right) \cdot \frac{\alpha_i^2}{\gamma_i^2} \cdot \frac{\gamma_i^2}{\alpha_i^2} \cdot \sum_{i=1}^{2} \frac{\gamma_i^4}{\gamma_i^2} \cdot \frac{\gamma_i^2}{\gamma_i^2}}
\]

(30)

when

\[
\gamma_i^4 = \frac{\gamma_i^4}{\gamma_i^4 + \gamma_i^4} \left( \frac{\alpha_i^2}{\gamma_i^2} \right) \frac{\gamma_i^4}{\gamma_i^2} + \frac{\gamma_i^4}{\gamma_i^4}
\]

Similarly the force ratio \( (P/P_c)^* \) may be expressed as a function of \( P/P_c \) of the first model (Fig. 4):

\[
(P/P_c)^* = \left( \frac{P}{P_c} \right) \left( 1 + \left( \gamma \frac{1^2}{c_j^*} e^* \right)^2 \right) \cdot \sum_{i=1}^{2} \gamma_i^4 \left( \frac{\gamma_i^2 - 2 \gamma_i^2}{\gamma_i^2 + \gamma_i^2} \right) \cdot \frac{\alpha_i^2}{\gamma_i^2} \cdot \frac{\gamma_i^2}{\alpha_i^2} \cdot \sum_{i=1}^{2} \frac{\gamma_i^4}{\gamma_i^2} \cdot \frac{\gamma_i^2}{\gamma_i^2}
\]

(40)

For small values of \( e^* \) it may be shown that:

\[
c_j = c_j \left( 1 + \left( \gamma \frac{1^2}{c_j^*} e^* \right)^2 \right)
\]

(41)

and

\[
(P/P_c) = \left( \frac{P}{P_c} \right) \left( 1 + \left( \gamma \frac{1^2}{c_j^*} e^* \right)^2 \right)
\]

(42)

When \( e^* \) is large, \( c_j \) and \( (P/P_c)^* \) become equal to their first model counterparts.
APPENDIX III. - REFERENCES


APPENDIX IV - NOTATION

The following symbols are used in this paper:

- $a$ - distance between elastic centers $EC$ and $SC$;
- $a_{ij}$ - perturbation coefficients;
- $C$ - damping matrix;
- $C_j$ - modal eccentricity amplification factor;
- $E$ - Young's Modulus;
- $e$ - static eccentricity;
- $e_d$ - dynamic eccentricity;
- $H$ - height of building;
- $h$ - story's height;
- $I$ - unit diagonal matrix;
- $I_x, I_y$ - second moment of area about $x$ and $y$ axes respectively;
- $I_{w}$ - second sectorial moment;
- $i$ - mass radius of giration;
- $K$ - stiffness matrix;
- $L_x, L_y$ - plan dimensions of building;
- $M$ - mass matrix;
- $N$ - number of reference levels;
- $P$ - equivalent coupled force;
- $P_c$ - uncoupled lateral force;
- $P_d$ - static eigenvalues;
- $q$ - modal amplitude
- $r_{x, y}$ - radii of gyration for $k$ and $s$ system respectively;
- $S$ - stiffness matrix
$S_a$ - acceleration spectrum;
$S_s$ - displacement spectrum;
$T$ - equivalent torsional moment;
$u_g$ - ground acceleration time history;
$v$ - displacement response vector;
$\alpha$ - uncoupled modal shapes ratio;
$\Gamma_j$ - modal participation factor;
$\Gamma_s$ - static participation matrix;
$\gamma$ - static eigenvector component;
$\varepsilon$ - perturbation factor;
$\varepsilon_j$ - correction coefficient;
$\Phi_j$ - modal shape matrix;
$\Phi_i$ - mode shape vector;
$K$ - stiffness matrix;
$\lambda$ - eigenvalue
$M$ - mass matrix;
$\omega$ - natural frequency;
$d\omega$ - damped natural frequency;
$\Psi_j$ - coupling shape matrix;
$\Psi_i$ - coupling shape vector;
$\zeta$ - damping ratio;

Subscripts
$L$ - lateral components;
$\alpha$ - rotational components;
$\sigma$ - uncoupled values.
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<table>
<thead>
<tr>
<th>Stories</th>
<th>Frame Properties ( 10^{-4} ) m</th>
<th>Wall Properties ( \times 10^{-4} ) m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( I_x )</td>
<td>( I_y )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0 - 4</td>
<td>71.458</td>
<td>142.916</td>
</tr>
<tr>
<td>4 - 10</td>
<td>57.214</td>
<td>114.427</td>
</tr>
<tr>
<td>10 - 16</td>
<td>26.042</td>
<td>52.084</td>
</tr>
<tr>
<td>16 - 20</td>
<td>18.984</td>
<td>37.968</td>
</tr>
</tbody>
</table>

Table I. - Stiffness Properties of Elements in Fig. 2

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Noncoupled</td>
<td>( \omega_{10} )</td>
<td>3.452</td>
<td>11.58</td>
<td>25.33</td>
<td>45.82</td>
<td>72.90</td>
</tr>
<tr>
<td></td>
<td>( \omega_{20} )</td>
<td>2.518</td>
<td>12.96</td>
<td>35.23</td>
<td>68.74</td>
<td>113.4</td>
</tr>
</tbody>
</table>

Table II. - Noncoupled Natural Frequencies for Example II
<table>
<thead>
<tr>
<th>Mode No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
<td>(9)</td>
<td>(10)</td>
<td>(11)</td>
</tr>
<tr>
<td>Natural Frequency (sec⁻¹)</td>
<td>exact</td>
<td>2.518</td>
<td>3.465</td>
<td>11.55</td>
<td>13.02</td>
<td>25.33</td>
<td>35.24</td>
<td>45.82</td>
<td>68.74</td>
<td>72.90</td>
</tr>
<tr>
<td></td>
<td>app.</td>
<td>2.518</td>
<td>3.465</td>
<td>11.55</td>
<td>13.02</td>
<td>25.33</td>
<td>35.24</td>
<td>45.82</td>
<td>68.74</td>
<td>72.90</td>
</tr>
<tr>
<td>Participation Factor</td>
<td>exact</td>
<td>1.164</td>
<td>2.053</td>
<td>-1.001</td>
<td>-0.360</td>
<td>0.613</td>
<td>0.282</td>
<td>0.459</td>
<td>0.207</td>
<td>-0.359</td>
</tr>
<tr>
<td></td>
<td>app.</td>
<td>1.165</td>
<td>2.052</td>
<td>-0.999</td>
<td>-0.369</td>
<td>0.615</td>
<td>0.286</td>
<td>0.458</td>
<td>0.209</td>
<td>-0.358</td>
</tr>
<tr>
<td>Error of approximated node's shapes in percent</td>
<td>mean error</td>
<td>0.52%</td>
<td>0.57%</td>
<td>1.96%</td>
<td>2.94%</td>
<td>6.69%</td>
<td>6.16%</td>
<td>3.71%</td>
<td>2.09%</td>
<td>11.11%</td>
</tr>
<tr>
<td></td>
<td>standard deviation</td>
<td>0.69%</td>
<td>0.79%</td>
<td>4.52%</td>
<td>9.24%</td>
<td>10.97%</td>
<td>17.28%</td>
<td>5.78%</td>
<td>2.54%</td>
<td>26.09%</td>
</tr>
</tbody>
</table>

**TABLE III.** - Dynamic Properties of Structures in Fig. 2

<table>
<thead>
<tr>
<th>Pair No. (j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
</tr>
<tr>
<td>( C' )</td>
<td>exact</td>
<td>11.74</td>
<td>10.85</td>
<td>11.03</td>
<td>11.94</td>
</tr>
<tr>
<td></td>
<td>app.</td>
<td>11.76</td>
<td>11.06</td>
<td>11.99</td>
<td>12.07</td>
</tr>
<tr>
<td>( P' )</td>
<td>exact</td>
<td>0.973</td>
<td>1.075</td>
<td>1.046</td>
<td>1.039</td>
</tr>
<tr>
<td></td>
<td>app.</td>
<td>0.974</td>
<td>1.075</td>
<td>1.046</td>
<td>1.039</td>
</tr>
</tbody>
</table>

**TABLE IV.** - Amplification Factors and Lateral Force Ratios for Structure in Fig. 2.
Fig. 3
Fig. 4
Amplification Factor $C_j^k = (e_{d_j}^n/e_j^n)^k$

Fig. 6