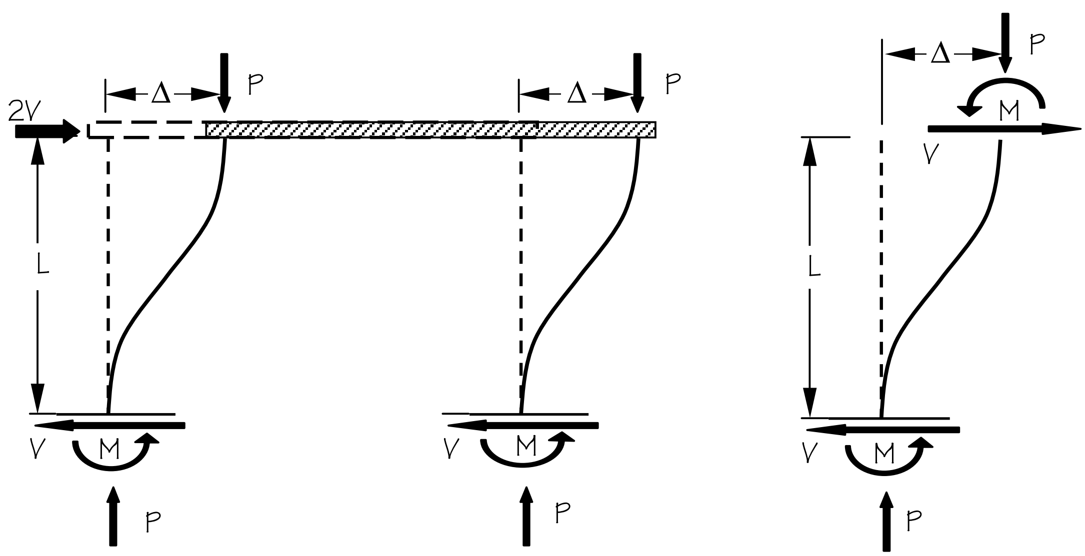
# SECTION 2
# LITERATURE REVIEW

## 2.1 General

The concept of P-Δ effects under static loading can be illustrated using the Single-Degree-Of-Freedom (SDOF) structure shown in figure 2-1. In this figure, 2P represents the force due to gravity acting on the mass lumped at the top of the structure, L is the structure height, 2V is the lateral force on the mass, and Δ is the horizontal displacement of the mass. As the structure sways by Δ under the effect of the lateral force, the product of P by Δ produces an additional moment at the base of each column which can be obtained by considering static equilibrium in the deformed configuration.




**(a) Portal frame with rigid beam** **(b) Single column removed from portal frame**

**FIGURE 2-1 Free Body Diagrams of Typical SDOF structure**

Many approximate methods have been developed (using equivalent lateral loads) for elastic static analysis to allow consideration of P-Δ effects using conventional equilibrium in the undeformed configuration. However, few studies have investigated P-Δ effects on yielding structures subjected to earthquake excitations. Some key parameters proposed by other researchers for the consideration of P-Δ effects in the seismic analysis of non-linear inelastic structures are described below, along with an overall view of the fundamental structural behavior.

## 2.2 Basic Theory

The relationship between the lateral force applied to the structure and its resulting displacement must be known in order to account for the P-Δ effect in design. Figure 2-2 displays a plot of the monotonically increasing lateral force, V, as a function of the lateral displacement, Δ. The solid line represents this relationship when P-Δ effects are ignored, while the dashed line represents the behavior when P-Δ effects are included.

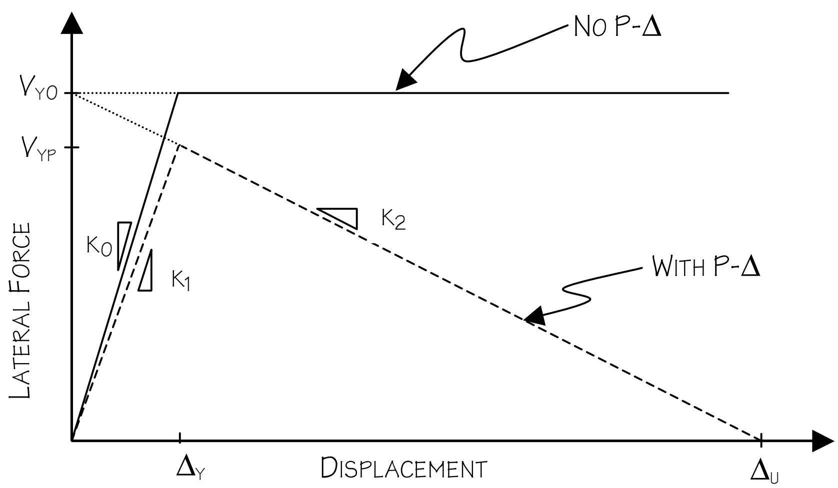



**FIGURE 2-2 Bilinear Lateral Force vs. Displacement model for SDOF structure**

If the typical SDOF structure considered in figure 2-1(a) is a single-bay portal frame with an infinitely rigid beam, the elastic lateral stiffness of each column within the frame, ignoring the P-Δ effect is given by:

$$K_o = {12 \cdot E \cdot I}/{L^3} \tag{2-1}$$

where:

E = Elastic Modulus of the material

*I* = Moment of inertia of the column section

*L* = Height of the column

For the bilinear elastic-perfectly plastic model shown, the ultimate lateral force, ignoring the P-Δ effect, that can be applied to that frame is reached when the plastic moment of the column, $M_p$, develops at the top and bottom of the column, and is given by:

$$V_{yo} = {2 \cdot M_p}/{L} \tag{2-2}$$

The corresponding yield displacement is:

$$\Delta_y = {V_{yo}}/{K_o} \tag{2-3}$$

Now, as shown in figure 2-1(b), considering P-Δ effects for a single column in the same frame, moment equilibrium gives:

$$2 \cdot M = VL + P\Delta \tag{2-4}$$

where $V$ is the lateral force at the top of the column

Rearranging (2-4), the lateral force, $V$, can be expressed as:

$$V = \frac{(2 \cdot M - P\Delta)}{L} = \frac{2 \cdot M}{L} - \frac{P\Delta}{L} = V_o - \frac{P\Delta}{L} \tag{2-5}$$

where $V_o$ is the lateral force that would be obtained ignoring the P-Δ effect.

Shown in figure 2-2, as a consequence of P-Δ effects seen in (2-5), $V$ decreases relative to $V_o$, as the displacement, Δ, increases. This equation can also be expressed as:

$$V = V_o - \frac{P\Delta}{L} = V_o - \theta K_o \Delta \tag{2-6}$$

where $\theta$ is the P-Δ stability factor given by:

$$\theta = \frac{P}{K_o L} \tag{2-7}$$

From (2-6), the elastic stiffness considering P-Δ, $K_1$, is therefore[1]:

[1] It has been suggested (Sivaselvan and Reinhorn, UB private communication, 2001) that better results could be obtained if (2-8) was replaced by: $K_1 = K_o \cdot \left[ \frac{p^3/3}{\tan(p) - p} \right]$ where $p = \frac{L}{2} \cdot \sqrt{\frac{P}{EI}}$. However, this potential

$$K_1 = K_o(1-\theta) \quad (2\text{-}8)$$

Similarly, the lateral force at which the system, including P-Δ effects, yields, $V_{yp}$, is:

$$V_{yp} = V_{yo}(1-\theta) \quad (2\text{-}9)$$

When elastic-perfectly plastic material properties are assumed for the idealized frame described earlier, lateral force $V_o$ in (2-6) remains constant in the post-elastic region of the force-displacement graph as the plastic moment, $M_p$, is developed. However, when P-Δ effects are considered, the corresponding lateral force versus displacement curve exhibits a negative slope past the yield point, with a stiffness of:

$$K_2 = -\theta \cdot K_o \quad (2\text{-}10)$$

as shown in figure 2-2.

Therefore, the monotonic bilinear force-displacement response of a SDOF structure, including P-Δ effects, can be summarized as follows:

$$V = \begin{cases} K_1 \cdot \Delta & if : \Delta \le \Delta_y \\ V_{yo} + K_2 \cdot \Delta & if : \Delta > \Delta_y \end{cases} \quad (2\text{-}11)$$

Another useful term in the characterization of P-Δ effects is the ratio of post-elastic to elastic stiffness, known as the stiffness ratio, *r*, and given by:

$$\mathrm{r} = \frac{\mathrm{K}_2}{\mathrm{K}_1} = \frac{\alpha - \theta}{1-\theta} \quad (2\text{-}12)$$

where $\alpha \cdot \mathrm{K}_o$ is the stiffness (in absence of stability effects) of the strain hardening segment of a bilinear elastic-plastic material model. In this study, as shown in figure 2-2, the value of α=0.0 is considered.

The concept of ductility is often used when discussing inelastic structural behavior. The displacement ductility of a structure, μ, is defined as:

$$\mu = \Delta / \Delta_y \quad (2\text{-}13)$$

improvement has not been quantified in this study as this recent development was brought to the authors' attention shortly before the publication of this report.

The ultimate displacement of the structure is designated as $\Delta_u$, as shown on figure 2-2, the point at which the negative-slope, post-elastic lateral strength curve intersects the displacement axis. This theoretically implies that for any additional lateral displacement, lateral instability develops (i.e. lateral strength becomes negative for any additional positive displacement). The ductility at ultimate displacement, $\Delta_u$, is known as the static stability limit, $\mu_s$, and derived using the relationships given in (2-3) and (2-10):

$$V = V_{yo} + K_2 \cdot \Delta_u = 0$$

$$V_{yo} - \theta \cdot K_o \cdot \Delta_u = 0$$

$$V_{yo} = K_o \cdot \Delta_y = \theta \cdot K_o \cdot \Delta_u$$

$$1/\theta = \frac{\Delta_u}{\Delta_y} = \mu_s \qquad \text{(2-14a)}$$

which can also be written in terms of the stiffness ratio using (2-12):

$$\mu_s = 1 - 1/r \qquad \text{(2-14b)}$$

## 2.3 Hysteresis Center Curve Concept

MacRae and Kawashima (1993) proposed the "hysteresis centre curve" (HCC) concept for analyzing the stability of general hysteresis loops. For the case of a bilinear system assumed for the analysis in this research, the HCC is a line parallel to the secondary stiffness that passes through the origin of the force-displacement space as pictured in figure 2-3.

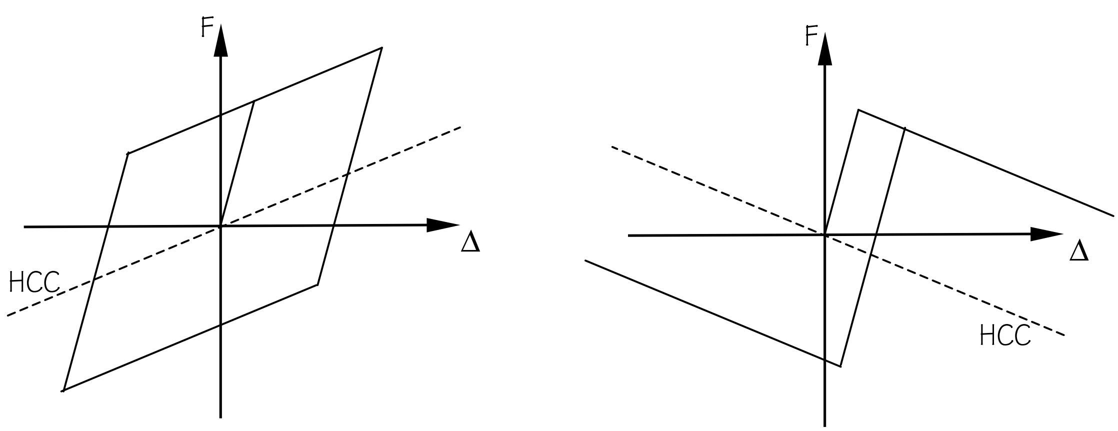



**FIGURE 2-3 Force vs. Displacement Behavior for a dynamically (a) stable and (b) unstable bilinear system**

The HCC characterizes a structure as follows: If the secondary stiffness, $K_2$, is positive as shown in figure 2-3(a) the system is considered stable and after the structure yields, will tend to return towards the point of zero displacement upon repeated reverse cyclic yielding during ground shaking. However, if $K_2$ is negative as shown in (b) of the same figure, the structure is deemed to be dynamically unstable.

The dynamically unstable system will tend to drift in a given direction once yielding has started. This results in large cumulative residual displacements and low cyclic energy absorption. Due to these large displacements, structures may be difficult to straighten and may perform poorly in a subsequent earthquake.

Some types of structures, such as reinforced concrete piers, will have a hysteresis loop which changes as the loading is occurring. This causes the point at which the HCC crosses the line of zero force to change from the initial location, even if the system response is stable.

## 2.4 Use of Amplification Factors to account for P-Δ Effects

Bernal (1987) investigated dynamic P-Δ effects in elastic and inelastic systems through the use of amplification factors. The ratio between displacement spectra with and without gravity effects represents the amplification spectrum that will amplify the elastic displacement for design.

Inelastic P-Δ effects were generated using four ground motions for a series of time history analyses. Damping was held constant at 5 percent of critical while the target displacement ductility and stability coefficient were both varied, from one to six and zero to 0.2, respectively, providing a total of 192 amplification spectra.

In the course of Bernal's parametric study, the target ductility was made to satisfy a maximum limit equal to $0.4\mu_s$, based on the requirement that the structure must remain fit to resist the factored gravity load following the inelastic response. In addition, the derivation assumed that the post-earthquake permanent deformation was equal to the maximum response ductility. Based on a statistical analysis of the 192 spectra generated in the study, the following expression for the amplification factor was proposed:

$$\alpha = \frac{1+\beta\theta}{1-\theta} \tag{2-15}$$

where regression analysis for the mean amplification yielded:

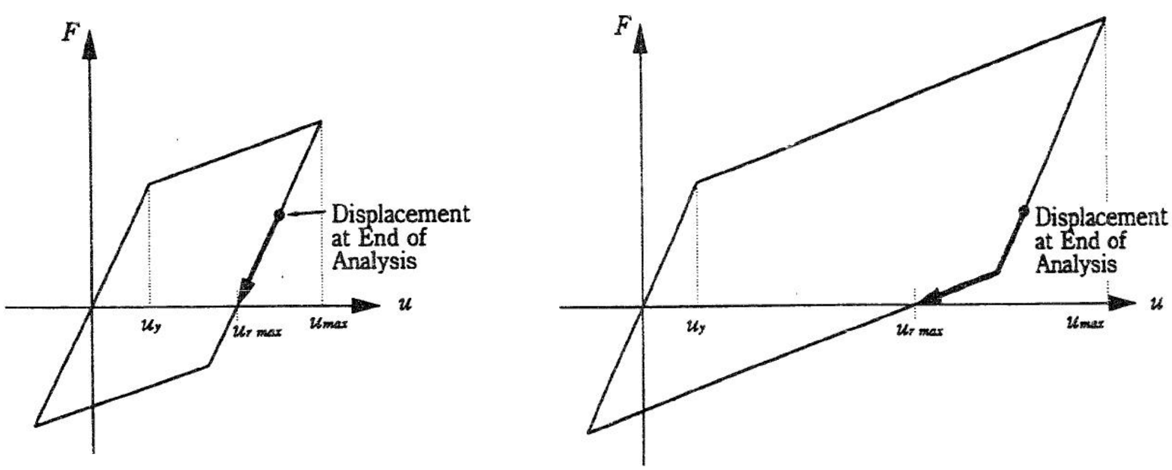
$$\beta = 1.87 \cdot (\mu - 1) \tag{2-16}$$

## 2.5 Residual Displacement Ratio Response Spectrum

Kawashima et al. (1996) proposed a technique for characterizing residual displacements of a structure following a seismic event, known as the residual displacement ratio, $S_{RDR}$. A SDOF oscillator with bilinear hysteretic restoring force was analyzed over a range of natural periods, stiffness ratios, and ductility for a number of ground motion time histories. Residual displacement ratio response spectra were designated as the plots of $S_{RDR}$ against the above-mentioned factors.

Three parameters are necessary to calculate the maximum residual displacement of the SDOF bilinear oscillator: the stiffness factor, r, the displacement ductility, μ, and the yield displacement, $\Delta_y$. The maximum residual displacement, illustrated in figure 2-4, is given by:

$$u_{r\,max} = \begin{cases} (\mu-1)(1-r)\Delta_y \xrightarrow{IF} r(\mu-1) < 1 \\ \left(\frac{1-r}{r}\right)\Delta_y \xrightarrow[IF]{} r(\mu-1) \geq 1 \end{cases} \tag{2-17}$$




(a) r(μ-1)<1 (b) r(μ-1)≥1

**FIGURE 2-4 Maximum Residual Displacement (Kawashima et al. 1996)**

The authors proposed the residual displacement ratio defined as:

$$S_{RDR} = \left| {u_r}/{u_{r\,max}} \right| \tag{2-18}$$

where $u_r$ is the residual displacement of the structure.

The stiffness ratio, r, was found to be the most significant factor controlling the residual displacement response ratio $S_{RDR}$. Figure 2-5 shows the mean value of $S_{RDR}$ versus $r$, as proposed by the authors for use in seismic design based on analyses of 63 components of ground motion. This proposed spectrum can be used for any structures where the hysteresis loop shape may be reasonably approximated by a bilinear model.

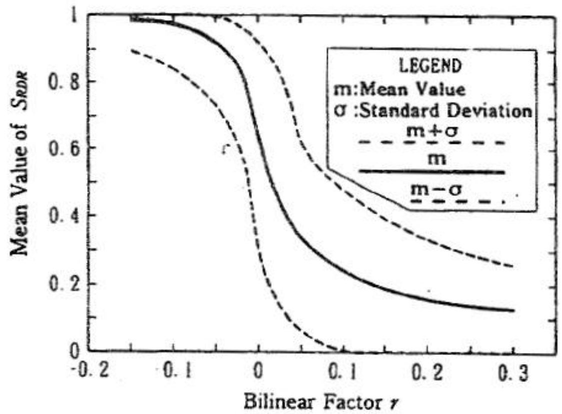



**FIGURE 2-5 Residual Displacement Ratio Response Spectra $S_{RDR}$ vs. Bilinear Factor *r* as Proposed for Design (Kawashima et al. 1996)**