Chapter 2 – Modeling and Similitude

2.1 Fundamental Theory

2.1.1 Physical Phenomenon and Its Dimension
2.1.1.1 Physical Phenomenon and Physical Variable
2.1.1.2 Unit and Dimension of Physical Phenomena
2.1.1.3 Similarity among Physical Phenomena
2.1.2 Determination of Pi-Terms
2.1.2.1 By Means of Laws in Physics
By Means of Governing Equations
Functions of pi-terms
By Dimensional Analysis
Dimension matrix.

2.2 Applications of Similitude Theory

2.2.1 Theory of Modeling
2.2.1.1 Type of Similarity
Geometric similarity
Distorted Similitude
Dissimilar Similitude
2.2.1.2. Types of Models
True, all pi-terms are the same
Adequate
Subsystems
2.2.2 Scaling Factor for Experimental Models
2.2.2.1 Design of Model Test
2.2.2.2.Type of Scale Factors
2.2.2.3 Scale Rules
2.2.2.4 Artificial Mass Simulation
2.3 Concluding Remarks

In this chapter we shall focus on one of the most fundamental concepts to design a model, namely, to scale down a prototype (full scaled). The major topics can be listed in the following:

1. Fundamental Theory of Similitude:

- Basic concepts of similitude and modeling.
- Basic theorems addressing the following questions:
  If two physical phenomena are similar, what can be expected? (1st theorem),
  Under what conditions are two physical phenomena similar? (3rd theorem)
  Essentially, how many terms do we need to determine a physical phenomenon? (2nd theorem)
- How to determine scaling factors, how to determine pi-terms
  By laws in Physics
  By governing equations
  By dimensional analysis

2. Applications in Dimensionless Analysis

- How to use dimensionless analysis to study an unknown physical phenomenon
- How to design a model test

When we carry out a test, we often use models instead of full-scaled objects. The reasons for using models are:
- They are easy to build and install in a laboratory.
- They can be affordably built.
- They make testing easier.
- They foster better understanding of local and global responses & behavior.
- They are accurate in elastic range.
- Although it may be difficult to model all details, many times there is no alternative to use models.

To deal with a model, we need the following theory:

2.1 Fundamental Theory

One of the aforementioned techniques for designing a test setup which will allow a better chance for a successful experiment is to make the test setup adjustable.

Adjustable means that an experimental procedure, a test setup should be flexible. It is understandable that the less constraint within a setup, the more flexibility we can have.

For example, if we try to visualize the Hook's law

$$F = k\,x \qquad (2.1\text{-}1a)$$

With virtually unlimited choice of materials, we can choose any available Young's modulus, any kind of cross section and length of specimen, any feasible loading. After all of these, only one parameter, say the deformation x is constrained.

If we are limited to choose a steel bar with given cross section and length, then we definitely find the experiments may be very difficult to carry out. Maybe the specimen is large and we do not have a large enough loading machine. Maybe it is too long that we don't have enough space, etc. In this case, our choices are limited and the test is difficult to adjust.

Generally speaking, a given experiment has limited number of parameters or variables. The fewer unknown variables we face, the smaller the number of equations we have to establish mathematically. In the following, we are going to introduce an important theory, one of the fundamental theories in experimental mechanics, the theory of similitude. It will not only let us determine the necessary number of those parameters, but will tell us how to make a test adjustable in general.

It can also give us a powerful tool to scale down (sometimes, maybe, scale up) a test model.

Yet it can also prove to be another tool to understand, to analyze, or to characterize those parameters or variables.

2.1.1 The Physical Phenomenon and its Dimension

2.1.1.1 Physical Phenomenon and Physical Variable

But what are those parameters or variables in the first place? To be specific, what are the above mentioned force f, spring constant k, deformation x?

It is well known that those variables are used to describe certain *physical phenomena*. For instance, the character f is used to describe a certain action of one object opposed to another one, called force, expressed in units, such as "newton" or "pound". The corresponding physical phenomenon is to put a load on a, say a structure, which will cause it to deform to a certain degree. The entire action is a physical phenomenon, which not only involves the force f, but also, a deformation of the structure, possibly some acceleration gain of a free body etc., which can be measured by the quantity x.

So, f and x are called physical variables, or physical quantities, in some textbooks.

2.1.1.2 Unit and Dimension of Physical Phenomena

In the following, to denote *Unit* we use the symbol ()
To denote *Dimension* we use the symbol [ ]

We have two kinds of dimensions:

Dependent dimensions

For example, In the SI system, the dimension of a force = [M L^2 T^{-1}]; and

Independent Dimensions, which are listed as follows

Dimension	SI unit	British unit
Mass	Kilogram (kg)	Pound mass (lbm)
Length	Meter (m)	Foot (ft)
Time	Second (s)	Second (s)

In ceratin cases we may also have additional independent dimensions, such as

Temperature Kelvin (K) Rankine degree (^{0} F)
Electric current Ampere (A) Ampere (A)

The basic dimensions in the Mass-Length-Time, MLT system, is called *SI system*
Or we can have alternative Force-Length-Time, FLT system, or called *British system*.
For example, in the relation between force and displacement,

F = k x

There exists 3 basic dimensions as follows:

$$[M\ L^1\ T^{-2}] = [M\ L\ T^{-2}]\ [L] \qquad (2.1\text{-}1b)$$

However, in another type of relationship, we have only one, namely

$$A = w\ h \quad [L^2\ ] = [L] \times [L] \qquad (2.1\text{-}2)$$

According to the pi-theorem (which will be discussed later on) the pi-term of

A = w h

is zero, which is referred to as the *Degenerated system.*

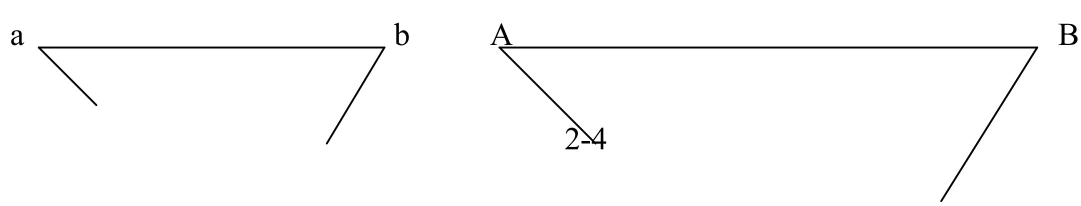
In the third equation, the dry friction, we have

$$F_c = \mu\ N \qquad (2.1\text{-}3)$$

Here the dimensionless quantity μ is treated as a pi-term. It is not degenerated

2.1.1.3 Similarity among Physical Phenomena

The essence of "similar" is proportional.
For example, let us examine the following two similar figures abcd and ABCD,



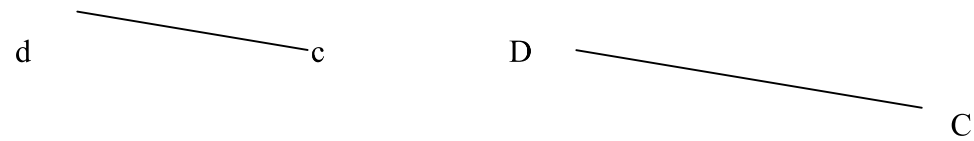


Figure 2.1-1 Geometry Similar

It is seen that

ab:AB = bc: BC = cd: CD = da : DA

Generally, we have three basic kind of similarity

1) Time intervals, similarity
2) Force, similarity
3) Velocity, similarity

An important condition of modeling, (reduced scaling, or scaling down, from a full scaled prototype to down scaled model), is that physical phenomena must be similar.
We haven't yet defined what does we mean by similarity of physical phenomena. This omission is intentional and will be explained after we introduce three basic theorems of similitude.

First Theorem (Bertrand 1848)

Two physical phenomena are similar, provided the similarity ratio is unity.

Or

If two physical phenomena are similar, their corresponding quantities must also be similar

Second Theorem (Vaschy, 1982, Riabouchinsky 1911 Buckingham 1914)

It is also called the Buckingham's Pi-theorem

The Second Theorey of Similitude

The Second Theorey of Similitude or the π-*-Theorem,* says that *if the equation*

$$E(q_1 , q_2 , q_3 \ldots q_n) = 0 \qquad (2.1\text{-}4)$$

is complete, the solution has the form

$$S(\pi_1 , \pi_2 , \pi_3 \ldots \pi_{n-k}) = 0 \qquad (2.1\text{-}5)$$

where the π - terms are independent products of the parameters q_1, q_2, etc., and are dimensionless in the fundamental dimensions. Here n is the total number of physical quantities and k is the number of fundamental dimensions.

In many cases, we have

$$k = 3 \quad (2.1\text{-}6)$$

Generally, we can referred to a dimensionless variable to be a pi-term. For example, the damping ratio ξ, the friction coefficient μ, etc.

The beauty of the π -Theorem is that the number of π -terms in equation (2.1-5) is less than that of the q parameters. Note that, the π -terms are actually dimensionless physical quantities. In order to make a model similar to a prototype, according to the First Theorem, the π - terms must be similar. It is understandable that in practice, the more π -terms we must deal with, the more difficult to conduct a test. In this sense, the Second Theorem releases the constraint of k physical quantities.

On the one hand, it makes the test easier to perform with k terms less. However, it still requires the rest n - k conditions to be satisfied. This is, by using the subscripts m and p to denote the model and prototype, the relationship with respect to π -terms can be written as

$$S_p(\pi_{1p}, \pi_{2p}, \pi_{3p} \ldots \pi_{(n-k)\,p}) = 0 \quad (2.1\text{-}7)$$

and

$$S_m(\pi_{1m}, \pi_{2m}, \pi_{3m} \ldots \pi_{(n-k)\,m}) = 0 \quad (2.1\text{-}8)$$

where

$$\begin{aligned} \pi_{1p} &= \pi_{1m} \\ \pi_{1p} &= \pi_{1m} \\ &\ldots\ldots \\ \pi_{(n-k)\,p} &= \pi_{(n-k)\,m} \end{aligned} \quad (2.1\text{–}9)$$

Note that, the inverse of π -Theorem is also true. That is, if a system is reduced from the form described by equation (2.1-4) into the form described by (2.1-5) with p π -terms, and the number of fundamental dimensions of the physical phenomenon is k, then the number n of physical quantities of the physical phenomenon is p + k. Namely,

$$n = p + k \quad (2.1\text{-}10)$$

In order to have some insight for further discussion of the similitude of subsystems, we briefly present the procedure for proving the π -Theorem.

Suppose the dimension of the first k terms in equation (2.1-4) are $[a_1]$, $[a_2]$ ... $[a_k]$ and

$[a_2] = A_2$ ... $[a_k] = A_k$.

where $A_{(.)}$'s are the specific units of each fundamental dimension.

It is known that the dimensions of rest n -k terms can be represented by the power function of the dimensions of the first k terms, (see Baker et al 1972, for instance), that is,

$$[a_{k+1}] = f_{k+1}\left([a_1], [a_2] \ldots [a_k]\right) = A_1^{P_1} A_2^{P_2} \ldots A_k^{P_k}$$
....
$$[a_n] = f_n\left([a_1], [a_2] \ldots [a_k]\right) = A_1^{r_1} A_2^{r_2} \ldots A_k^{r_k}$$

If the units of the physical quantities of the first k terms are respectively decreased by the factors

$c_1 , c_2 ,\ldots c_k$

then the values of each physical quantities in equation (2.1-4) will be increased by the same set of factors $c_1 , c_2 ,\ldots c_k$ respectively. Denote the new values of the fundamental physical quantities after the increase by

$$q_1 = c_1\, q_1$$
$$q_2 = c_2\, q_2$$

...

$$q_k = c_k\, q_k$$

We then have values of the newly derived physical quantities as follows:

$$q_{k+1} = c_1^{P_1} c_2^{P_2} \ldots c_k^{P_k}\, q_{k+1}$$
...
$$q_n = c_1^{r_1} c_2^{r_2} \ldots c_k^{r_k}\, q_n$$

Therefore, equation (2.1-5) will become

$$E(q'_1 , q'_2 , q'_3 \ldots q'_n) = 0 \qquad (2.1\text{-}11)$$

With the above notations, we can rewrite equation (2.1-11) into

$$E(c_1 q_1, c_2 q_2, \ldots, c_k q_k, (c_1^{P_1} c_2^{P_2} \ldots c_k^{P_k}) q_{k+1}, \ldots (c_1^{r_1} c_2^{r_2} \ldots c_k^{r_k})\, q_n) = 0 \qquad (2.1\text{-}12)$$

Denoting

$c_1 = 1/q_1$, $c_2 = 1/q_2$, ..., $c_k = 1/q_k$, (2.1-14)

yields

$E(1, 1, ...,1, q_{k+1} /\{q_1^{P_1}q_2^{P_2}\ldots q_k^{P_k}\},..., q_n/\{ q_1^{r_1}q_2^{r_2}\ldots q_k^{r_k}\}) = 0$ (2.1-15)

Illuminate the k 1's and let

$q_{k+1} /\{q_1^{P_1}q_2^{P_2}\ldots q_k^{P_k}\} = \pi_1$

...

$q_n/\{ q_1^{r_1}q_2^{r_2}\ldots q_k^{r_k}\} = \pi_{n-k}$

we finally have

$E(\pi_1 , \pi_2 , \pi_3 \ldots \pi_{n-k}) = 0$

From this derivation, we realize that, the mapping from equation (2.1-5) to (2.1-11) is the key transformation that cancels the fundamental dimensions from the **entire equation**. This is done, in fact, by a linear multiplication

$L_L = c_1 c_2 \ldots c_k = (q_1 q_2 \ldots q_k)^{-1}$

on both sides of equation (2.1-5). This is

$E = L_L E$ (2.1-16)

In the above discussion, we use bold face to mark the words "entire equation" because:

i) Equation (2.1-4) or its solution (2.1-5) is a single equation, instead of a group of equations. And, in the above proof, we keep the form of single equation. A generic linear transformation may decouple a matrix equation into several scalar equations. For reasons, some of these scalar equations can be neglected and only the rest are used only. This kind of neglection is **not** involved in the statement of the second theory of similitude, **nor** involved in its proof.

ii) There exists another way to cancel the fundamental dimension performed within local terms. For example, with the FLT system, equation (1c) can be rewritten as

$F_I [F] - M [F L^{-1} T^2] X'' [L T^{-2}] = 0$ (2.1-17)

It can be seen that, the dimensions [L] and [T] appear only in the term MX" but not in F. It can be canceled within the local term M X". Therefore, globally, it is not necessary to have the operation described in equation (2.1-17) to get rid of [L] and [T]. This kind of cancellation is equivalent to insert a unitary term {1} with the "dimension" [$[A][A]^{-1}$], where A is a proper dimension. That is,

$\{ 1 [[A] [A]^{-1}])$ (2.1-18)

Third Theorem (Kupnhyeb, 1930)

If two physical phenomena of the same kind have all the same pi-terms, then they are similar.

This theorem is often referred to as the *inverse theorem* of the similitude, which indicates the necessary condition of that two physical phenomena of the same kind are similar. The necessary condition is that all the pi-terms should be identical. Generally speaking, two physical phenomena of the same kind means their governing equations have the same dimensions and unit. For example, the vibration problems of a prototype straucture and a model are of the same type. Whereas the vibration problem of a structure with mass-damping-stiff and of a circuity with coil-resistor-capacitor are of different physical phenomenon.

2.1.2 Determination of Pi-Terms

Generally speaking: the theory of similitude consists of two major portions: What is similarity, and what are the applications in experimental mechanics. In the following, let us first discuss how to find these pi-terms.

2.1.2.1 By Means of Laws in Physics

we can find the pi-terms by using various laws in physics, The general procedure is listed as follows.

a. Find related laws, (note that, some of these laws are essential, others may be unnecessary)
b. Write them in dimensional forms
c. Reduce the dimensional form into dimensionless term, which is original pi-term
(Since some of the laws are inessential, we have to reduce the number of the “original” pi-terms)
d. Obtain required pi-term(s) from the original pi-terms
For example, π_{o1}, π_{o2}, π_{o3} But π_{o2} is unnecessary, obtain $\pi = \pi_{o1} / \pi_{o3}$
e. Derive required scaling factors

Example 1:

A 5 lb. turkey needs 5 hours to cook fully, how many hours is needed for a 10 lb. turkey under the same temperature T ? is that 10 hrs?

a. First, we have the corresponding laws in physics, they are:
(1) Heat transfer to the turkey, governed by

$$Q_k = k\ A\ T/l\ t \qquad (2.1\text{-}19)$$

(2) Heat contain inside the turkey, with the formula

$$Q_c = c\ \rho\ V\ T \qquad (2.1\text{-}20)$$

where K is the coefficient of heat conduct; A is the area of the turkey; T is the temperature; l stands for length and thickness; t is the time; c denotes the specific heat; ρ is the mass density; and V denotes volume.

b. Second, we rewrite these two laws in dimensional form, namely

$$Q_k = k\ A\ T/l\ t \rightarrow k\ l\ T\ t \quad \text{since } A: l^2 \qquad (2.1\text{-}21)$$
$$Q_c = c\ \rho\ V\ T \rightarrow c\ \rho\ l^3\ T \quad \text{since } V : l^3 \qquad (2.1\text{-}22)$$

where the symbol → stands for changing a regular formula into dimensionless form; and the symbol (x):(y) explains the basic demensions of (y) in the quantity (x).

c. Thirdly, we can reduce the above dimensionless form into original pi-terms, that is,

$$\pi_{o1} = Q_k / (\ k\ l\ T\ t\ ) \qquad (2.1\text{-}23)$$
$$\pi_{o2} = Q_c /(\ c\ \rho\ l^3\ T\ ) \qquad (2.1\text{-}24)$$

Note that, only the first pi-term contains the time t. However, both the above contain some quantities we cannot control, namely Q_k and Q_c. In order to find the required pi-trem, we should get rid off them.

d. Forthly, in order to get rid off Q_k and Q_c by finding a new pi-term, we can divide Q_k from Q_c, that is,

$$\pi = Q_k / Q_c = (\ k\ l\ T\ t\ ) /(\ c\ \rho\ l^3\ T\ ) \rightarrow (\ k\ t\ ) /(\ c\ \rho\ l^2\ ) \qquad (2.1\text{-}25)$$

e. Finally, let us find the scaling factors of the time according to the third theorem of similitude, that is,

$$\pi_p = \pi_m$$

In this way, we can write,

$$(\ k_p\ t_p\ ) /(\ c_p\ \rho_p\ l_p^{\ 2}\ ) = (\ k_m\ t_m\ ) /(\ c_m\ \rho_m\ l_m^{\ 2}\ ) \qquad (2.1\text{-}26a)$$

Usually we do not have to write the subscript of the prototype for simplisity, namely,

$\pi = \pi_m$

Then, we have

$(k\, t) /(c\, \rho\, l^2) = (k_m\, t_m) /(c_m\, \rho_m\, l_m{}^2)$ (2.1-26b)

Note that, for given same oven, same "material" of turkey , we should have

$k = k_m$; $c = c_m$; and $\rho = \rho_m$

therefore, we can write

$t_m / t = (l_m / l)^2$ (2.1-27)

Since

$(l_m / l) = (W_m / W)^{1/3}$

we finally have

$t_m = t\ [W_m / W)^{1/3}]^2 = 5\ [10 / 5)^{1/3}]^2 = 8$ (hr) (2.1-28)

Note that, it is not 10 hr!

Example 2:

Design a model beam to study the time duration needed for free-decay vibration of a prototype. (Is that $(t_m / t) = (l_m / l)^{1/2}$?)

a. Laws in physics are listed as follows:

(1) spring force $\sigma = E\, \varepsilon$ (2.1-29)

(2) initial force $dF = dm \times a$ (2.1-30)

(3) damping $dU = dV\, C\, \sigma_m{}^3 = dV\, C\, \sigma^3$ (2.1-31)

b. Dimnentional analysis is performed

From $\sigma \rightarrow E\, \varepsilon$, we further have $F \rightarrow l^2 E\, \varepsilon$ since $\sigma \rightarrow F / l^2$
From $F \rightarrow m\, a F \rightarrow \rho\, l^4 /t^2$ since $m \rightarrow \rho\, V$, and $V \rightarrow l^3$; as well as $a \rightarrow l / t^2$
From $U \rightarrow V\, C\, \sigma^3$, we further have $F \rightarrow l^2 /(C)^{1/2}$ since $V \rightarrow l^3$;
From $\sigma \rightarrow F / l^2$; we further have $U \rightarrow F\, l$

c. Original pi-terms are thus determined as follows:

$\pi_{o1} = F /(l^2 E \varepsilon) = F /(l^2 E)$ (2.1-32)
$\pi_{o2} = F / (\rho l^4 /t^2)$ (2.1-33)
$\pi_{o3} = F / (l^2 /(C)^{½})$ (2.1-34)

Note that, only the second one has time t. Therefore, only one pi-term is needed

d. We can either use

$\pi_1 = \pi_{o1} /\pi_{o2} = (\rho l^4 /t^2)/ (l^2 E) = (\rho l^2)/ (t^2 E)$ (2.1-35a)

or use

$\pi_2 = \pi_{o1} /\pi_{o3} = (\rho l^4 /t^2)/ (l^2 /(C)^{½}) = [\rho l^2 (C)^{½})]/ (t^2)$ (2.1-35b)

to find new pi-terms

e. To scale factors of the time according to the third theorem, $\pi = \pi_m$, we can write

$(t_m^2 / t^2) = (l_m^2 / l^2)$ (2.1-36)

Furthermore we write

$(t_m / t) = (l_m / l)$ (2.1-37)

Note that, the ratio of time is directly proportional to the ratio of length, instead of

$(t_m / t) = (l_m / l)^{1/2}$

Again, some of the laws are unnecessary to use, which is the 3rd one, in this particular case. Note that both the first and the second pi-terms imply $\sigma_m = \sigma$ whereas the third one does not.

Unnecessary Laws

In many cases, we do not introduce unnecessary laws. Let us now use the above example to examine if adding another law(s), then what will happen?

Add an obvious law

$W = mg$ (2.1-38)

which implys $F \rightarrow \rho g l^3$ since $m \rightarrow \rho l^3$

We then have additional pi-term

$\pi_{o4} = F / (\rho g l^3)$ (2.1-39)

Thus, we will have

$\pi_3 = \pi_{o1} / \pi_{o4} = (\rho\, l^4 / t^2) / (\rho\, g\, l^3) = l / (t^2)$

note that here $c_g = 1$ for g is constant

Therefore, from $\pi = \pi_m$, we can write

$(t_m^2 / t^2) = (l_m / l)$ which implys
$(t_m / t) = (l_m / l)^{1/2}$

This result may contradict to the aforementioned

$(t_m / t) = (l_m / l)$

Or the only possible solution is that to let

$(l_m / l) = 1;$ (2.1-40)

In this case, the model is nothing but exactly the prototype itself. That becomes a travail case. This indicates that we shouldn't consider the fourth law at all.

However, sometimes, we have to consider additional laws to complete correct analysis. When and why should we consider additional law(s)? Let us examine the following example, first.

Test for base isolation of bubble shock is conceptually shown in figure 2.1-1,: A boat with deck isolation of spring constant k is excited by a bubble. To conduct a model test, we need to determine the spring constant or the natural frequency. We virtually have two different cases. 1) When we consider the isolation system itself without any necessary knowledge of the propagating of the bubble shock, the gravity constant is quite important. 2) When we study the wave propagation, however, the gravity constant becomes unnecessary law. If we attempt to add this law, we will make the studies very complicated.

Freq. = $(g/l)^{1/2} / 2\pi$ | Freq. = ?

$(t_m / t) = (l_m / l)^{1/2}$ | $(t_m / t) = (l_m / l)$

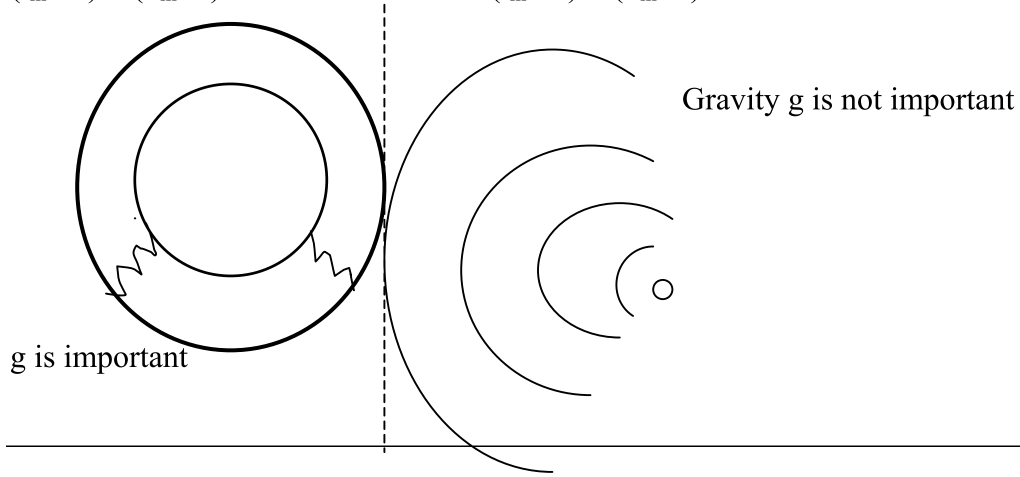


Figure 2.1-1 Isolation of bubble shock

2.1.2.2 By Means of Governing Equations

We can also find the pi-terms by means of governing equations. To explain this method, let us take a look at the example of above-mentioned free decay vibration of a beam.
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Example:

In the aforementioned example of free decay vibration of a beam, we have the physical quantities and corresponding dimensions listed as follows:

displacement	y [L]
mass	m [$FL^{-1} T^2$]
damping	c [$F L^{-1}T$]
stiffness	k [$F L^{-1}$]
initial disp.	y_o [L]
initial vel.	v_o [$L T^{-1}$]
time:	t [T]

It is seen that the number of pi-terms is

$$7 - 3 \text{ (since } k = 3) = 4. \quad (2.1\text{-}41)$$

We can have the following two methods to determine the pi-terms. In each case, we briefly list the procedure of analyses.

Method (I) Similarity Transformation

a. First, let us write down two equations for the prototype and the model, that is,

$$m\, d^2 y / d^2 t + c\, d y / d t + k\, y = 0 \quad (2.1\text{-}42)$$

$$m_m\, d^2 y_m / d^2 t_m + c_m\, d y_m / d t_m + k_m\, y_m = 0 \quad (2.1\text{-}43)$$

b. Secondly, reduce them into one equation, by transferring (2.1-43) into

$$(C_m C_y / C_t^2)\, m\, d^2 y / d^2 t + (C_c C_y / C_t)\, c\, d y / d t + (C_k C_y)\, k\, y = 0 \quad (2.1\text{-}44)$$

c. Similarity ratio is then found by Comparing (2.1-44) and (2.1-42) in the following:

$$(C_m C_y / C_t^2) = (C_c C_y / C_t) = (C_k C_y) \quad (2.1\text{-}45)$$

That is, from

$(C_m C_y / C_t^2) = (C_c C_y / C_t)$,

we have

$(C_c C_t / C_m) = 1$ (2.1-46)

Similarly, from $(C_m C_y / C_t^2) = (C_k C_y)$,

we have

$(C_k C_t^2 / C_m) = 1$ (2.1-47)

Note that, in the above, we have only two equations to determine two terms. We thus need another two, since from the above calculation (2.1-40) : 7-3 = 4; we should have totally four terms. Now, from initial condition:

$C_y = C_{yo}$

we have

$C_y / C_{yo} = 1$ (2.1-48)

And from the second initial condition:

$C_y / C_t = C_{vo}$

we have

$C_{vo} C_t / C_y = 1$

or $C_{vo} C_t / C_{yo} = 1$ (2.1-49)

d. Finally, we can find the pi-terms from equations (2.1-46) through (2.1-49), namely
From (2.1-46) $c\, t/m$ = constant = dimensionless
From (2.1-47) $k\, t^2/m$ = constant
From (2.1-48) y/y_o = constant
From (2.1-49) $v_o\, t/ y_o$ = constant

Since y is the variable, equation in pi-terms becomes

$y/y_o = f\,(ct/m\, ,\, k\, t^2/m,\, v_o\, t/ y_o)$ (2.1-50)

Method (II) Integration Analogue

a. First again, let us write down two equations for the prototype and the model

$$m\, d^2 y / d^2 t + c\, d y / d t + k\, y = 0 \qquad (2.1\text{-}51)$$

b. Find a term as common denominator for example, the 3rd term k y
We then have two dimensionless terms

$(m\, d^2 y / d^2 t)/(k\, y)$ and

$$(c\, d y / d t)/(k\, y) = 0 \qquad (2.1\text{-}52)$$

c. Derivatives replaced by ratios

For example, use y/t to replace dy /dt
Use y/t^2 to replace $d^2 y / d^2 t$ etc.
So, (2.1-51) becomes m/(kt) = constant =dimensionless term pi
So, (2.1-52) becomes c/(kt) = constant = another dimensionless term pi

Again, we need another two terms

From $y = y_o$ we have y / y_o = constant =dimensionless term pi
From $dy/dt = v_o$ we have $y / (v_o t)$ = constant or $(v_o t)/ y_o$ = constant.
Then we have

$$y/y_o = f\,(m/(kt), c/(kt), v_o t / y_o) \qquad (2.1\text{-}53)$$

Comparing the one obtained through the first method described in equation (2.1-50), we find that they look different. However, these two equation should posses the same essence. In the next section, we shall attempt to explain this phenomenon.

2.1.2.3. Functions of pi-Terms

We shall see that the functions of pi –terms are still pi-terms, which will therefore explain the question asked above. That is, we can derive one dimensionless equation from another one.

Suppose we have $\pi_1, \pi_2, \ldots \pi_r$ The following are also pi-terms:
1) $(\pi_i)^{(\alpha i)}$
2) $(\pi_1)^{(\alpha 1)} (\pi_2)^{(\alpha 2)} \ldots (\pi_r)^{(\alpha r)}$
3) $(\pi_1)^{(\alpha 1)} + (\pi_2)^{(\alpha 2)} + \ldots + (\pi_r)^{(\alpha r)}$
4) $\pi_i + \alpha_i$
5) $\alpha_i \pi_i$

2.1.2.2 **By Dimensional Analysis**

We can also determine the pi-terms by the dimensional analysis. In the following, let us use examples to describe the procedure.

Dimension matrix.

Using same example mentioned above of the case of m-c-k free decay vibration, we can manage the unit of those physical quantities vs. their corresponding basic dimensions as follows:

	y	m	c	k	v_o	y_o	t
F	0	1	1	1	0	0	0
L	1	-1	-1	-1	1	1	0
T	0	2	1	0	-1	0	1

Where the first row is the list of physical quantities. The second row begins with a basic dimension F. The entry at the cross point of F and y is zero, which means that the dimension of quantity y does not contain force. The second row begins with a basic dimension L. The entry at the cross point of L and y is 1, which means that the dimension of quantity y contains the length to the power one. The third row begins with a basic dimension T. The entry at the cross point of T and y is zero, which means that the dimension of quantity y does not contain time.

From the above list, we can further have the *dimension matrix* D as follows. It is seen that the entry of D is chosen according to the corresponding list mentioned above. That is,

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \quad (2.1\text{-}54)$$

Matrix D can provide useful information. First, we have

$$k = \text{rank}\,(D) \quad (2.1\text{-}55)$$

In the above example

$$\text{rank}\,(D) = 3$$

since at least one of the minor determinant of D with order 3 is of full rank.

The dimension matrix can also be used to find pi –terms. Adding the power of each physical quantity, i.e. a_1, a_2 , a_3 , a_4 , a_5 , a_6 and a_7 on the top of the dimension matrix, we have the follows:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
	y	m	c	k	v_o	y_o	t
F	0	1	1	1	0	0	0

L	1	-1	-1	-1	1	1	0
T	0	2	1	0	-1	0	1

Note that a pi-term is a function of these physical parameters, that is,

$$\pi = y^{a_1} m^{a_2} c^{a_3} k^{a_4} v_o^{a_5} \; y_o^{a_6} \; t^{a_7} \qquad (2.1\text{-}56)$$

So, we will have three equations obtained from (2.1-56), that is,

From the force F, which must have a zero power, thus,

$$a_2 + a_3 + a_4 = 0 \qquad (2.1\text{-}57)$$

Similarly, from length L, we have

$$a_1 - a_2 - a_3 - a_4 + a_5 + a_6 = 0 \qquad (2.1\text{-}58)$$

And, from time T, we have

$$2a_2 + a_3 - a_5 + a_7 = 0 \qquad (2.1\text{-}59)$$

The above equations hold because the pi-term π is dimensionless, that is, $\pi \rightarrow \pi^0$

To sovle the three equations with seven variables, we must assume some of them known parameters. For example, let us assume a_4 , a_5 , a_6 and a_7 are known. The above equations become

$$a_1 = - a_5 - a_6 \qquad (2.1\text{-}60)$$
$$a_2 = a_4 + a_5 - a_7 \qquad (2.1\text{-}61)$$
$$a_3 = -2a_4 - a_5 + a_7 \qquad (2.1\text{-}62)$$

The numbers of our pi-terms must be 7 – 3 = 4; We assume a certain sets of values of the four parameters a_4 , a_5 , a_6 and a_7 and apparently we need 4 sets. Let us assume set 1)

$a_4 = 1$ and $a_5 = a_6 = a_7 = 0$.

So, we have $a_1 = 0$, $a_2 = 1$, $a_3 = -2$; And therefore

$$\pi_1 = mk/c^2 \qquad (2.1\text{-}63)$$

Similarly, assume set 2)

$a_5 = 1$ $a_3 = a_6 = a_7 = 0$.

So, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$; And

$$\pi_2 = mv_o/yc \qquad (2.1\text{-}64)$$

Now, assume set 3)

$a_6 = 1$ $a_3 = a_4 = a_7 = 0$. So, $a_1 = -1$, $a_2 = 0$, $a_3 = 0$; Thus,

$$\pi_3 = y_o/y \qquad (2.1\text{-}65)$$

Finally, assume set 4)

$a_7 = 1$ $a_3 = a_4 = a_5 = 0$. So, $a_1 = 0$, $a_2 = -1$, $a_3 = 1$; And,

$$\pi_4 = ct/m \qquad (2.1\text{-}66)$$

So, with the help of equations (2.1-63) through (2.1-66), we can have

$$y_o/y = f\,(mk/c^2\,,\ mv_o/yc\,,\ ct/m\,) \qquad (2.1\text{-}67)$$

The above equation has mv_o/yc as its hidden variable, which is not convenient to use. We then can change it into to

$$y_o/y = f\,(mk/c^2\,,\ mv_o/y_o\,c\,,\ ct/m\,) \qquad (2.1\text{-}68)$$

by using the function of pi-terms described above.

2.1.2.3 Basic Characteristics of a Group of Pi-Terms

Basic characteristics of a group of pi-terms $\pi_1, \pi_2, \ldots \pi_r$ that are a complete set to describe a physical phenomenon with n physical quantities

1) each term is dimensionless
2) $\pi_1, \pi_2, \ldots \pi_r$ are independent
3) $r = n - k$, where k is the number of basic dimensions

How to determine the number k is important, on the surface, we check the rank of dimension matrix. However, it is not guaranteed that some of the physical quantities will not be missing. In general, the disadvantages of dimensional analysis are that

1) May miss some important physical quantities
2) May add some inessential physical quantities
3) Difficult to distinguish physical quantities with the same dimension, which have different physical meanings, for example, pressure, stress, Young's modulus, etc. with the same dimension $[FL^{-2}]$.
4) May ignore physical quantities with zero dimension, such as friction coefficient
5) Possible difficulty in finding some physical constant which have dimensions, such as gravity g
6) Among pi-terms, some are important, some are not, not easy to determine which is more important.

2.2 Applications of Similitude Theory

The second port of this chapter is one of the important applications of the similitude theory, the modeling of prototype structures. Different from mathematical models, the modeling in this chapter focusing on the issue of scaling as well as related topics.

2.2.1 Theory of Modeling

Generally, a mathematical model that represents realistic structures unveils abstract relationship of certain quantities of the structure, such as forces vs. deformations. On the other hand, a physical model that represents the abstract relationship exhibits quantities by real-world models. In this case, since the physical model is often down scaled, the issue of selecting correct scaling factor becomes very important. Correct scaling factors are determined through similitude analysis.

2.2.1.1 Type of Similarity

To apply the theory of similitude analysis, let us first examine several useful types of similarity:

1. Geometric similarity

A model and the prototype having same shape and behavior geometrically as shown in figure 2.2-1 promise *geometric similarity*.

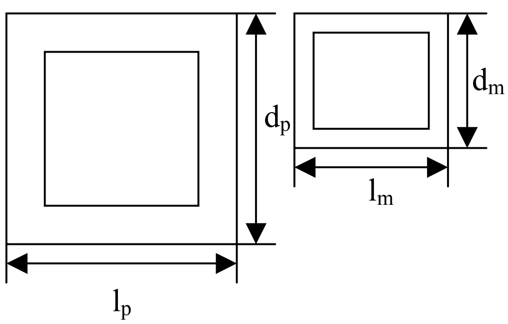


Figure 2.2-1 Geometric similarity

From figure 2.2-1, it is seen that the following hold:

$$l_p / l_m = d_p / d_m \qquad (2.2\text{-}1)$$

and

$$A_p / A_m = (l_p / l_m)^2 = (d_p / d_m)^2 \qquad (2.2\text{-}2)$$

Denote the scaling factor of the length to be,

$\lambda_l = l_p / l_m$, (2.2-3)

and the scaling factor of the area to be

$\lambda_A = \lambda_l^2$ (2.2-4)

In the following, we use Greak letter $\lambda_{(.)}$ to denote the scaling factor of quantity (.).

2. Distorted Similitude

We may also have distorted similitude, if two or more scale factors of the same quantity are used.

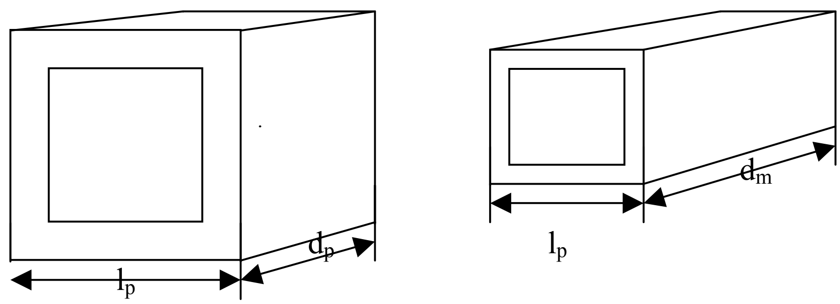


Figure 2.2-2 Distorted similarity

It is seen that, in this case of distorted similarity,

$\lambda_l = l_p / l_m \neq \lambda_d = d_p / d_m$ (2.2-5)

3. Dissimilar Similitude

The third type of similarity is the dissimilar similitude. In this case, we have no resemblance between the model and the prototype.

2.2.1.2. Type of Models

In our laboratory, we can have three different type of models, they are
1. *True model*, in this case, all pi-terms are the same.
2. *Adequate model*, in the case, the major pi-terms are indentical.
3. *Subsystems*, in this case, a port of the model is a subsystem of the original model.

We shall discuss these models in detail in a later section.

2.2.2 Scaling Factor for Experimental Models

In this section, we start to introduce the detail concept of scaling factor and the method to determine these scaling factors.

2.2.2.1 Design of Model test

First of all, let us examine the procedure of how to design a model test. Figure 2.2-3 shows the block diagram of the procedure.

It is seen that from the above diagram, in order to design a proper model, the issue of selecting correct scaling factor is a necessary step. Without determination of suitable factors, we will have no models. Is seems that we can first select the factor of length and then according similitude law, we can figure out the rest of selection on other scaling factors. In real tests, however, we often design a model by an iterative or trial-and-error procedure, for the possibility of improper selection on some factors that are not realizable.

2.2.2.2.Type of Scale Factors

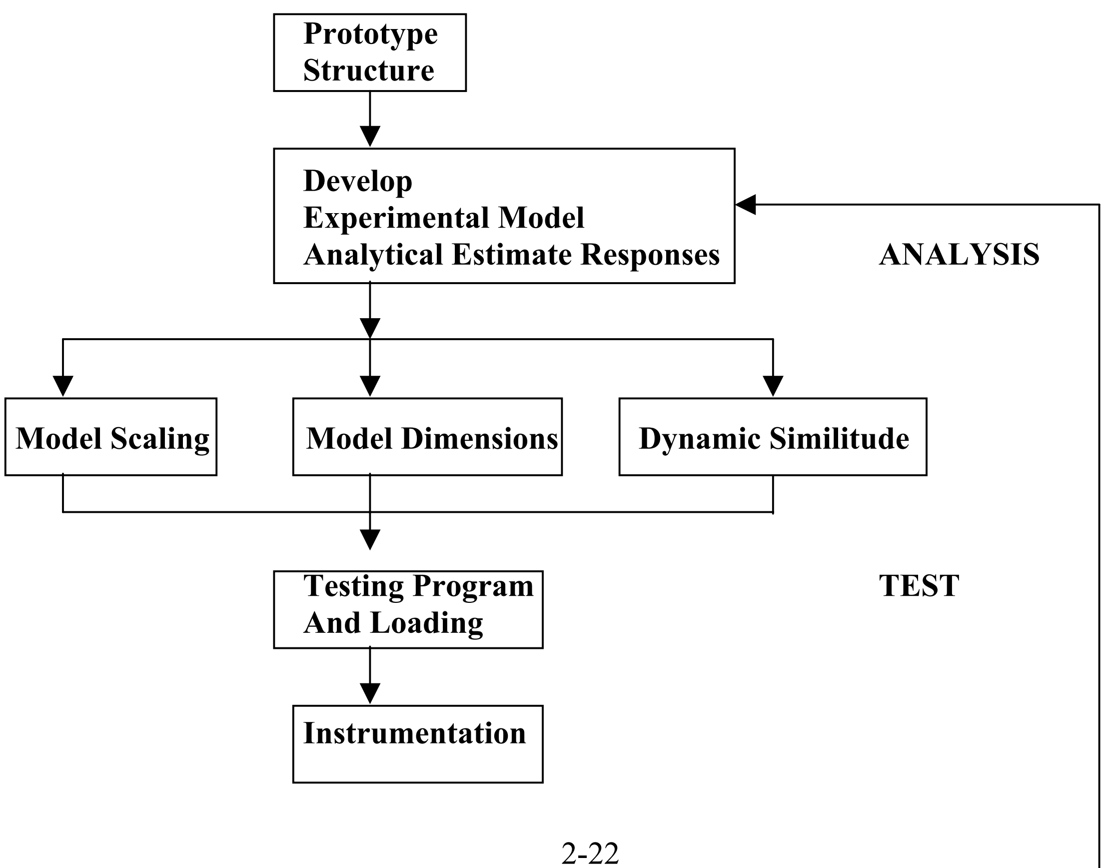
As mentioned above, the scaling factor $\lambda_{(.)}$ is defined to be

$$\lambda_{(.)} = (.)_p / (.)_m \qquad (2.2\text{-}6)$$

For example, the

Length scale

$$\lambda_l = \text{distance in the prototype/distance in the model} \qquad (2.2\text{-}7)$$



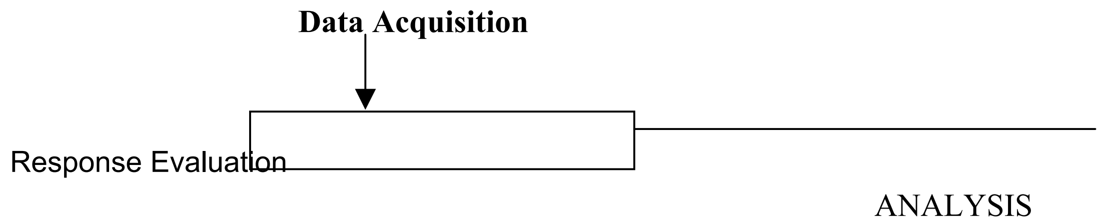


Figure 2.2-3 Design of model test

Often, when we design a model, we select the length scale first and then determine the rest factors accordingly. We can also have the following factors that are selected first:

Force scale

λ_F = forces applied to prototype/forces applied to the model (2.2-8)

Mass scale

λ_F = mass of prototype/mass of the model (2.2-9)

Area scale

λ_F =area of prototype/area of the model (2.2-10)

2.2.2.3 Scale Rules

To determine all the similitude ratios, the dimensionless equation must be satisfied, i.e. the requirement of that the corresponding pi-terms must be identical should be satisfied
In the ideal case, all the physical quantities satisfy the similitude law. In reality, we may be forced to alter some quantities' similitude or ignore unimportant parameters. Certain conditions can limit our choices, including the following:

Using the same material, the mass density, λ_ρ , Young's modulus, λ_E , viscosity, λ_v, etc. are forced to be identical, that is,

$\lambda_\rho = 1$; (2.2-11)
$\lambda_E = 1$; (2.2-12)
$\lambda_v = 1$, (2.2-13)

In the same gravity filed, the gravitation constant g will remain the same, that is , the scaling factor of the gravity, λ_g, remains unchanged,

$\lambda_g = 1$ (2.2-14)

Since the gravity is sessentially of acccleration, we may also have the scaling factor of acceleration λ_a to be

$\lambda_a = \lambda_g = 1$ (2.2-15)

In the same media, the sound speed, and the wave propagation speed, etc. should have the same value. In the case, we have the following scale factor of sound speed, λ_{sp}, and/or wave propagation speed, λ_{wp},

$\lambda_{sp} = 1$ (2.2-16)
$\lambda_{wp} = 1$ (2.2-17)

In the following, let us use an example to show how to determine these scaling factors with a certain given factors.

Example: let the factor of length $\lambda_l = 3$, while the factors of Young's modulus, aceelerations and gravity unchanged, $\lambda_E = 1$, $\lambda_a = \lambda_g = 1$; i.e.

$\lambda_l = 3 \rightarrow \lambda_{width} = 3$; (2.2-18)

and note that

$Width_p = 3 \text{ x } Width_m$ (2.2-19)

Cross section area	$A = L^2$	$\lambda_A = \lambda_l^2 = 9$	(2.2-20)
Moment of inertia	$I = L^4$	$\lambda_I = \lambda_l^4 = 81$	(2.2-21)
Volume	$V = L^3$	$\lambda_V = \lambda_l^3 = 27$	(2.2-22)
Density	$\rho = m/V = F/aV = E/aL$	$\lambda_\rho = \lambda_E / (\lambda_a \lambda_l) = 1/3$	(2.2-23)
Time (period)	$t = (L/a)^{1/2}$	$\lambda_t = (\lambda_l / \lambda_a)^{1/2} = 1.73$	(2.2-24)
Velocity	$v = a\, t$	$\lambda_v = \lambda_a (\lambda_l / \lambda_a)^{1/2} = (\lambda_l \lambda_a)^{1/2} = 1.73$	(2.2-25)
Mass	$m = \rho\, V$	$\lambda_m = \lambda_\rho \lambda_V = \lambda_\rho \lambda_l^3 = 27$	(2.2-26)
Force	$F = m\, a$	$\lambda_F = (\lambda_\rho \lambda_V) \lambda_a = \lambda_E \lambda_l^2 = 9$	(2.2-27)
Stress		$\lambda_\sigma = \lambda_E = 1$	(2.2-28)
Strain	$\varepsilon = \sigma/E$	$\lambda_\varepsilon = \lambda_E / \lambda_E = 1$	(2.2-29)

Impulse	$I = F\, t$	$\lambda_I = \lambda_E \lambda_l^2 (\lambda_l / \lambda_a)^{1/2} = \lambda_l^3 (\lambda_E / \lambda_\rho)^{1/2} = 27$	(2.2-30)
Frequency	$\omega = (k/m)^{1/2}$	$\lambda_\omega = [(\lambda_F / \lambda_l)/(\lambda_\rho / \lambda_l^3)]^{1/2} = 1/\lambda_l (\lambda_E/\lambda_\rho)^{1/2} = 0.58$	(2.2-31)
Strain Rate	$\varepsilon' = d\varepsilon/dt$	$\lambda_{\varepsilon/} = \lambda_E / \lambda_t = (\lambda_l / \lambda_a)^{1/2} = 1.73$	(2.2-32)
Damping Ratio		$\lambda_\xi = 1$	(2.2-33)
Energy	$e = F\, L$	$\lambda_e = \lambda_F \lambda_l = \lambda_E \lambda_l^2 \lambda_l = \lambda_E \lambda_l^3 = 27$	(2.2-34)

2.2.2.4 Artificial Mass Simulation

Suppose a model with the same material, then the similitude ratio of mass density is fixed that is

$\lambda_\rho = 1$

However, it may be required that

$\lambda_\rho = \lambda_l^{-1}$

to satisfy the pi-theorem, so, artificial mass must be added

That is, the mass similitude must be altered while other quantities are preserved. The reason of why we have to use and how to use the artificial mass are shown as follows:

We must have the scaling factor of mass in a model test,

$\lambda_M = \lambda_\rho \ \lambda_l^{-3}$

We may have two chices:

1) $\lambda_M = \lambda_l^{-3}$ if $\lambda_\rho = 1$ (real world provided)

and

2) $\lambda_M = \lambda_l^{-2}$ if $\lambda_\rho = \lambda_l^{-1}$ (model design required)

The added mass can be then determined by the following procedure:

$$m_m = m_p \lambda_l^{-2} = m_p (\lambda_l^{-2} - \lambda_l^{-3} + \lambda_l^{-3}) = m_p \lambda_l^{-3} + m_p (\lambda_l^{-2} - \lambda_l^{-3}) = m_m' + \Delta m \quad (2.2\text{-}35)$$

where

$$m_m' = m_p \lambda_l^{-3} \quad (2.2\text{-}36)$$

and

$$\Delta m = m_p (\lambda_l^{-2} - \lambda_l^{-3}) = m_p (\lambda_l - 1) \lambda_l^{-3} \quad (2.2\text{-}37)$$

Or

$$\Delta m = m_m' (\lambda_l - 1) \quad (2.2\text{-}38)$$

By means of the above formulae, we can calculate the necessary mass to be added to the testing model.

2.3 Concluding Remarks

1. Scaled model studies are the only practical way to experimentally test civil engineering structures, which are extremely large
2. The results of model studies can be extrapolated for elastic and for ideal plastic behavior (difficult, however)
3. Model studies serve to validate analytical tools, provide data for parametric studies, explore behavior of complex systems, and validate pilot implementation
4. Design of models and their construction require extreme care. Small variations can be critical for interpretation of results.
5. Nonlinear structures can be studied using analytical models that are validated by scaled models.