Chapter 6
Energy Concepts in Earthquake Engineering
CONTENT

1. Introduction
2. Rain Flow Analogy
3. Energy Balance Equation
4. Examples of Energy Computation
1. Introduction

• Seismic energy formulation natural way to understand effect of supplemental damping and seismic isolation systems

• Main advantages of energy formulation:
  – replacement of vector quantities (displacements, velocities and accelerations) by scalar energy quantities
  – flow of energy quantities can be tracked during seismic response
2. Rain Flow Analogy

During seismic shaking
2. Rain Flow Analogy

At the end of seismic shaking

\[ V_{in} = V_d + V_h \]
3. Energy Balance Equation

• Derivation

The governing differential equations of motion for a general nonlinear MDOF system excited at the base by a horizontal translation from an earthquake ground motion are given in matrix form by:

\[
[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + \{F_r(t)\} = -[M]\{r\}\dot{x}_g(t) + \{F_s\}
\]

(3.2)

where:

• \([M]\) is the global mass matrix.

• \([C]\) is the global viscous damping matrix which accounts for all inherent velocity dependent energy dissipating mechanisms in the structure other than the inelastic hysteretic energy dissipated by the structural members. Note that these damping mechanisms are usually not velocity dependent, but are expressed in this way for mathematical convenience.

• \(\{x(t)\}\), \(\{\dot{x}(t)\}\) and \(\{x(t)\}\) are respectively the vectors of global accelerations, velocities and displacements relative to the moving base at time \(t\).

• \(\{F_r(t)\}\) is the vector of global nonlinear restoring forces at time \(t\) generated by the hysteretic characteristics of the structural elements.

• \(\{r\}\) is a vector coupling the directions of the ground motion input with the directions of the DOFs of the structure.

• \(\dot{x}_g(t)\) is the horizontal acceleration of the ground at time \(t\).

• \(\{F_s\}\) is the vector of global static loads applied to the structure prior to and maintained during the seismic excitation.
3. Energy Balance Equation

The formulation presented in Equation (3.2) is derived for equal excitation at all support points of the structure. For non-synchronous excitation at the different supports of the structure (multiple support excitation), the right hand side of Equation (3.2) is modified as presented in Chopra (2001).

The energy formulation is obtained by integrating the work done by each element in Equation (3.2) over an increment of global structural displacements \( \{dx\} \):

\[
\int \{dx\}^T [M] \{\ddot{x}(t)\} + \int \{dx\}^T [C] \{\dot{x}(t)\} + \int \{dx\}^T \{F_r(t)\} = -\int \{dx\}^T [M] \{r\} x_s(t) + \int \{dx\}^T \{F_s\}
\]

Recalling the differential relationships:

\[
\{dx(t)\} = \{x(t)\} dt \quad (3.4)
\]

\[
\{dx(t)\} = \{x(t)\} dt \quad (3.5)
\]

Using Equation (3.4), the first two terms on the left side of Equation (3.3) are first rewritten as:

\[
\int \{dx\}^T [M] \{\dddot{x}(t)\} = \int \{\dot{x}\}^T [M] \{\ddot{x}(t)\} dt
\]

\[
\int \{dx\}^T [C] \{\dot{x}(t)\} = \int \{x\}^T [C] \{x(t)\} dt
\]

where the integrals on the right hand side are taken over time.

Using Equations (3.4) and (3.5), the expressions obtained in Equation (3.6) are rewritten as:

\[
\int \{x\}^T [M] \{x(t)\} dt = \int \{x\}^T [M] \{dx(t)\}
\]

\[
\int \{\dot{x}\}^T [C] \{\dot{x}(t)\} dt = \int \{\dot{x}\}^T [C] \{dx(t)\}
\]
3. Energy Balance Equation

The energy formulation is finally written as:

\[ \int \{ x(t) \}^T [M] \{ dx(t) \} + \int \{ x(t) \}^T [C] \{ dx(t) \} + \int \{ dx \}^T \{ F_e(t) \} \]
\[ = -\int \{ dx \}^T [M] \{ r \} x_g(t) + \int \{ dx \}^T \{ F_s \} \]  

(3.8)

The first term of Equation (3.8), can be integrated directly and expressed as:

\[ \int \{ x(t) \}^T [M] \{ dx(t) \} = \frac{1}{2} \{ x(t) \}^T [M] \{ x(t) \} \]  

(3.9)

Based on Equation (3.8) the energy balance equation is defined as:

\[ E'_k(t) + E_{vd}(t) + E_a(t) = E'_m(t) + E_a(t) \]  

(3.10)

where:

- \( E'_k(t) \) is defined as the relative kinetic energy at time t:
  \[ E'_k(t) = \frac{1}{2} \{ x(t) \}^T [M] \{ x(t) \} \]  

(3.11)

- \( E_{vd}(t) \) is the energy dissipated by viscous damping from the beginning of the record up to time t:
  \[ E_{vd}(t) = \int \{ x(t) \}^T [C] \{ dx(t) \} \]  

(3.12)

- \( E_a(t) \) is the absorbed energy from the beginning of the record up to time t:
  \[ E_a(t) = \int \{ dx \}^T \{ F_e(t) \} \]  

(3.13)

- \( E'_m(t) \) is the relative input energy from the beginning of the record up to time t:
  \[ E'_m(t) = -\int \{ dx \}^T [M] \{ r \} x_g(t) \]  

(3.14)

- \( E_a(t) \) is the work done by static loads applied before and maintained during the seismic excitation from the moment of application of the forces up to time t:
  \[ E_a(t) = \int \{ dx \}^T \{ F_s \} \]  

(3.15)
3. Energy Balance Equation

The absorbed energy term $E_a(t)$ represents the total amount of energy that the structure has absorbed either through elastic straining or unrecoverable inelastic deformations of its elements. The peak absorbed energy during an earthquake represents the largest demand on structural members, and is expressed as the sum of two components:

$$E_a(t) = E_{es}(t) + E_h(t) \tag{3.16}$$

where $E_{es}(t)$ is the recoverable elastic strain energy at time $t$ and $E_h(t)$ is the energy dissipated through hysteretic damping of the structural elements up to time $t$, and depends on the hysteretic relation of each structural member. The definition of the elastic strain energy and hysteretic energy terms is further discussed in the following paragraph where the energy formulation is presented in discrete expressions.

Recalling Equation (3.4), it can be seen that the damping energy expressed by Equation (3.12) monotonically increases throughout the time-history, whereas the absorbed energy expressed by Equation (3.13) fluctuates while generally increasing. Referring to Equation (3.16), the fluctuations in the absorbed energy are caused by the elastic strain energy that is absorbed and then restored.
3. Energy Balance Equation

• Relative and absolute input energy
  – Equation (3.10) based on equivalent seismic load applied to a rigid based structure
    • i.e. “relative” energy formulation
  – Rigid body translation of structure not considered
  – “Absolute” energy formulation required
    • Can be derived from equations of motion
    • Relative energy formulation can be transformed into absolute energy formulation
3. Energy Balance Equation

To achieve this, the relative input energy $E_{in}^r(t)$ is first rewritten using the differential relations of Equation (3.4) as an integral over time:

$$ E_{in}^r(t) = -\int \{\dot{x}(t)\}^T [M] \{r\} \ddot{x}_g(t) dt $$

Equation (3.17) is then integrated by parts:

$$ E_{in}^r(t) = -\{\dot{x}(t)\}^T [M] \{r\} \dot{x}_g(t) + \int \{\ddot{x}(t)\}^T [M] \{r\} \dot{x}_g(t) dt $$

The relative acceleration vector $\{\ddot{x}(t)\}$ can be expressed in terms of the absolute acceleration vector $\{\ddot{x}_a(t)\}$ as:

$$ \{\ddot{x}(t)\} = \{\ddot{x}_a(t)\} - \{r\} \ddot{x}_g(t) $$

Substituting Equation (3.19) into Equation (3.18) yields:

$$ E_{in}^r(t) = -\{\dot{x}(t)\}^T [M] \{r\} \dot{x}_g(t) + \int \{\ddot{x}_a(t)\}^T [M] \{r\} \dot{x}_g(t) dt $$

$$ -\int \ddot{x}_g(t) \{r\}^T [M] \{r\} \dot{x}_g(t) dt $$
3. Energy Balance Equation

Using the differential relationships expressed in Equations (3.4) and (3.5) in terms of the ground displacement, velocity and accelerations:

\[
E_{in}(t) = -\{\dot{x}(t)\}^T[M]\{r\}\dot{x}_g(t) + \int \{\ddot{x}_a(t)\}^T[M]\{r\}d\dot{x}_g(t)
\]

\[
-\int\dot{x}_g(t)\{r\}^T[M]\{r\}d\dot{x}_g(t)
\]

The last term of Equation (3.21) can be directly integrated to yield:

\[
E_{in}(t) = -\{\dot{x}(t)\}^T[M]\{r\}\dot{x}_g(t) + \int \{\ddot{x}_a(t)\}^T[M]\{r\}d\dot{x}_g(t)
\]

\[
-\frac{1}{2}\dot{x}_g(t)\{r\}^T[M]\{r\}\dot{x}_g(t)
\]

Introducing Equation (3.22) into the energy balance equation (Equation (3.8)) results in:

\[
\frac{1}{2}\{\dot{x}(t)\}^T[M]\{\dot{x}(t)\} + \{\dot{x}(t)\}^T[M]\{r\}\dot{x}_g(t) + \frac{1}{2}\dot{x}_g(t)\{r\}^T[M]\{r\}\dot{x}_g(t)
\]

\[
+ E_{vd}(t) + E_a(t) = \int \{\ddot{x}_a(t)\}^T[M]\{r\}d\dot{x}_g(t) + E_{st}(t)
\]

Recognizing a perfect square in the first three terms of the left hand side, Equation (3.23) becomes:

\[
\frac{1}{2}\{\{\dot{x}(t)\} + \{r\}\dot{x}_g(t)\}^T[M]\{\{\dot{x}(t)\} + \{r\}\dot{x}_g(t)\}
\]

\[
+ E_{vd}(t) + E_a(t) = \int \{\ddot{x}_a(t)\}^T[M]\{r\}d\dot{x}_g(t) + E_{st}(t)
\]
3. Energy Balance Equation

Recalling Equation (3.19) and extending it to the relative velocity vector, we get:

\[ \{\dot{x}(t)\} = \{\dot{x}_a(t)\} - \{r\} \dot{x}_g(t) \]  \hspace{1cm} (3.25)

Substituting Equation (3.25) into Equation (3.24), we obtain the absolute energy formulation:

\[ E_k^a(t) + E_{vd}(t) + E_a(t) = E_{in}(t) + E_{st}(t) \]  \hspace{1cm} (3.26)

where:

- \( E_k^a(t) \) is defined as the absolute kinetic energy of the system at time \( t \):

\[ E_k^a(t) = \frac{1}{2} \{\dot{x}_a(t)\}^T [M] \{\dot{x}_a(t)\} \]  \hspace{1cm} (3.27)

- \( E_{in}(t) \) is defined as the absolute input energy of the system from the beginning of the record up to time \( t \):

\[ E_{in}^a(t) = \int \{\dot{x}_a(t)\}^T [M] \{r\} dx_g(t) \]  \hspace{1cm} (3.28)

and where \( E_{vd}(t) \), \( E_a(t) \) and \( E_{st}(t) \) have been previously defined in Equations (3.13) and (3.15) and remain unchanged in both relative and absolute energy formulations.
3. Energy Balance Equation

• Remarks on Relative and Absolute Energy Formulations
  – Absolute input energy = base shear * ground displacement: true physical meaning
  – Both formulations are mathematically equivalent
  – Sums of kinetic and input energy equal in both formulations
  – Ground displacement required in absolute formulation
  – Uang and Bertero (1990) showed that relative and absolute input energies equal for wide period range (0.1 sec to 5.0 sec)
3. Energy Balance Equation

- Discrete energy expressions
  - Required for time-integration schemes
  - Kinetic energy obtained directly at each time
  - Other energy quantities require time integration
  - e.g. Trapezoidal rule:

\[
E_{\text{vd}}(t) = E_{\text{vd}}(t-\Delta t) + \frac{1}{2}(\{\dot{x}(t-\Delta t)\} + \{\dot{x}(t)\})^T[C](\{x(t)\} - \{x(t-\Delta t)\})
\]

\[
E_a(t) = E_a(t-\Delta t) + \frac{1}{2}(\{x(t)\} - \{x(t-\Delta t)\})^T(\{F_r(t-\Delta t)\} + \{F_r(t)\})
\]

\[
E_{\text{in}}^r(t) = E_{\text{in}}^r(t-\Delta t)
\]

\[
\frac{1}{2}(\{x(t)\} - \{x(t-\Delta t)\})^T[M]\{r\} (\{\ddot{x}_g(t-\Delta t)\} + \{\ddot{x}_g(t)\})
\]

\[
E_{\text{in}}^q(t) = E_{\text{in}}^q(t-\Delta t)
\]

\[
\frac{1}{2}(\{\ddot{x}_a(t-\Delta t)\} + \{\ddot{x}_a(t)\})^T[M]\{r\} (\{x_g(t)\} - x_g\{(t-\Delta t)\})
\]
3. Energy Balance Equation

- Energy Balance
  - All energy terms computed individually
  - Energy Balance Error (EBE) can be calculated at each time-step
  - Tolerance can be set on EBE
  - EBE can be normalized as follows:

\[
EBE^r(t) = \left| \frac{E_{in}^r(t) + E_{st}^r(t) - E_k^r(t) - E_{vd}(t) - E_a(t)}{E_{in}^r(t)} \right|
\]  
(3.35)

for the relative energy formulation, and:

\[
EBE^a(t) = \left| \frac{E_{in}^a(t) + E_{st}^a(t) - E_k^a(t) - E_{vd}(t) - E_a(t)}{E_{in}^a(t)} \right|
\]  
(3.36)

for the absolute formulation.
4. Examples of Energy Computation

• Scope
  – Ensemble of six two-storey frames
  – Idealized as 2-DOF systems
  – Three frames incorporate supplemental damping and seismic isolation systems
  – Energy computation for various earthquake ground motions
4. Examples of Energy Computation

• Structural Models
  – Moment Resisting Frame (MRF)
  – Braced Moment Resisting Frame (BMRF)
    • Elastic braces added to MRF
  – Soft Storey Frame (SSF)
    • First floor braces removed from BMRF
  – Base Isolated Frame (BIF)
    • Laminated rubber bearings installed under BMRF
    • Bearings modeled as elastic springs
    • Supplemental DOF considered above bearings
  – Friction Damped Braced Frame (FDBF)
    • Friction dampers installed in braces of BMRF
    • Dampers modeled as Coulomb friction elements
  – Viscously Damped Braced Frame (VDBF)
    • Linear viscous dampers installed in braces of BMRF
    • Braces connected to dampers assumed rigid
    • Damping constant based on 23% damping in first mode
4. Examples of Energy Computation

- Structural Models

- 2% structural damping in each mode

Figure 3.3 Structures Considered in Energy Calculations
### 4. Examples of Energy Computation

- **Structural Models**

Table 3-1: Properties of Structures Considered in Energy Calculations

<table>
<thead>
<tr>
<th>Frame</th>
<th>Mass (kN-s²/m) Level 1</th>
<th>Mass (kN-s²/m) Level 2</th>
<th>Initial Lateral Stiffness (kN/m) Level 1</th>
<th>Initial Lateral Stiffness (kN/m) Level 2</th>
<th>Yield Shear (kN) Level 1</th>
<th>Yield Shear (kN) Level 2</th>
<th>Initial Natural Period (s) Mode 1</th>
<th>Initial Natural Period (s) Mode 2</th>
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<td>169000</td>
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</table>

*a* Isolation bearing lateral stiffness = 1960 kN/m

*b* Interstorey shear corresponding to the sliding of the friction dampers

*c* Interstorey shear corresponding to yielding of the frame structure

*d* Interstorey damping constant for viscous dampers $C_h = 640$ kN-s/m
4. Examples of Energy Computation

• Earthquake Ground Motions
  – Records:
    • 1940 Imperial Valley earthquake recorded at El Centro (S00E)
    • 1988 Saguenay earthquake recorded at Chicoutimi (TRAN)
    • 1977 Romania earthquake recorded at Budapest (North-South)
  – All records scaled to PGA of 0.50 g
  – First 15 seconds of each records considered
4. Examples of Energy Computation

• Earthquake Ground Motions

Figure 3.4 Accelerograms of Earthquake Ground Motions Scaled to 0.5g
4. Examples of Energy Computation

• Earthquake Ground Motions

![Graph showing spectral acceleration response](image)

**Figure 3.5 Absolute Acceleration Response Spectra of Earthquake Ground Motions**
4. Examples of Energy Computation

- Peak Transient Displacement Ductility Ratios

<table>
<thead>
<tr>
<th>Ground Motion</th>
<th>Structural System</th>
<th>Displacement Ductility, $\mu_\Delta$</th>
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4. Examples of Energy Computation

- Energy Time-Histories
4. Examples of Energy Computation

- Comparison of Relative Input Energy
4. Examples of Energy Computation

• Comparison of First Floor Absorbed Energy

![Graph showing energy absorption over time for different floor types under El Centro record.](image)

*Figure 3.13 First Floor Absorbed (Elastic Strain + Hysteretic) Energy Time-Histories under El Centro Record*
4. Examples of Energy Computation

• Fraction of Seismic Input Energy Absorbed
Questions/Discussions